Notes on the run of α

Alexey A. Vladimirov

This note are made in order to fix signs and solutions, and not forget about Landau pole. On the base of these note the `AlphaStrong.m` is build.

I. EVOLUTION EQUATION ON a_s

The QCD coupling is defined as

$$a_s = \frac{g^2}{(4\pi)^2}.$$

In the following we often drop subscript s. The evolution equation reads

$$\mu^2 \frac{da_s}{d\mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - .... \;, \tag{1.1}$$

where

$$\beta_0 = \frac{11}{3}C_A - \frac{2N_f}{3} > 0, \tag{1.2}$$

$$\beta_1 = 102 - \frac{38N_f}{3} \tag{1.3}$$

$$\beta_2 = \frac{2857}{2} - \frac{5033N_f}{18} + \frac{325N_f^2}{54}. \tag{1.4}$$

We also would introduce the following notation

$$B_i = \frac{\beta_i}{\beta_0}, \quad B_0 = 1. \tag{1.5}$$

We define Λ_{QCD} as the position of Landau pole, i.e.

$$a_s(\mu^2 > \Lambda^2) > 0, \qquad a_s(\mu^2 \to \Lambda^2) \to +\infty.$$

We do not care about smaller values of Λ.

II. LO SOLUTION

LO solution is simple and exact

$$a_s(\mu) = \frac{1}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} = \frac{1}{z}. \tag{2.1}$$

I.e.

$$z = \beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)$$

III. NLO SOLUTION

The NLO solution is already not so simple, therefore, often people use approximate solutions, see [1, 2]. The exact solution reads

$$a_s^{(2)}(\mu) = -\frac{1}{B_1}\left(1 + W_{-1}\Big[-\exp\left(-\frac{z}{B_1} - 1\right)\Big]\right)^{-1}, \tag{3.1}$$

where W_{-1} is the Lambert function.

There are exists to approximation to this solution. One is called iterative [1]

$$a_{it}^{(2)}(\mu) = \frac{1}{z + B_1 \ln\left(1 + \frac{z}{B_1}\right)}. \tag{3.2}$$

This solution works reasonably bad. The second iteration gives

$$a_{it2}^{(2)}(\mu) = \frac{1}{z + B_1 \ln\left(1 + \frac{z}{B_1} + \ln\left(1 + \frac{z}{B_1}\right)\right)}, \tag{3.3}$$

which works even better.

Another solution is the high-energy solution [3]

$$a_{HEP}^{(2)}(\mu) = \frac{1}{z}\left(1 - \frac{B_1}{z}\ln\left(\frac{z}{\beta_0}\right)\right). \tag{3.4}$$

This solution is much smoother at $\mu \to \Lambda$.

The deviations of solutions are given in the table (here $N_f = 3$)

$\frac{\mu}{\Lambda}$	$a_s^{(2)}$	$\Delta a_{it}^{(2)}$	$\Delta a_{it2}^{(2)}$	$\Delta a_{HEP}^{(2)}$
1.5	0.064	-26%	-7.5%	-156 %
2	0.044	-15%	-3.4%	-47 %
4	0.026	-6.8%	-1.05%	-8.3 %
10	0.017	-3.6%	-0.39%	-1.65 %
100	0.0097	-1.38%	-0.089%	0.062 %
1000	0.0068	-0.75%	-0.035%	0.093 %

One can see that the HEP solution is rather bad at low scale, by better then first iteration at higher energy. The second iteration is much better then both.

IV. NNLO SOLUTION

The solution at NNLO cannot be found in exact form.[1] However one can easily invert it numerically.

[1] We have

$$z = \int \frac{\rho^2 d\rho}{(\rho + q_1)(\rho + q_2)} = \frac{1}{q_1 - q_2}\int \rho d\rho \left(\frac{q_1}{\rho + q_1} - \frac{q_2}{\rho + q_2}\right), \tag{4.1}$$

where $\rho = a^{-1}$ and

$$q_1 = \frac{1}{2}\left(B_1 + \sqrt{B_1^2 - 4B_2}\right), \qquad q_2 = \frac{1}{2}\left(B_1 - \sqrt{B_1^2 - 4B_2}\right), \qquad B_2 = \frac{\beta_2}{\beta_0}. \tag{4.2}$$

Note, that $B_1^2 - 4B_2 < 0$, therefore, these roots are complex conjugated to each other. Integrating we present this equation in the convenient form

$$q_1\left(z - \rho + q_1 \ln\left(\frac{\rho}{q_1} + 1\right)\right) = q_2\left(z - \rho + q_2 \ln\left(\frac{\rho}{q_2} + 1\right)\right), \tag{4.3}$$

where we add an arbitrary constant such that $\rho(z \to 0) \to 0$. A solution of this transcendental equation is unknown. Let us solve an auxiliary equation

$$q_i\left(z - \rho + q_i \ln\left(\frac{\rho}{q_i} + 1\right)\right) = y. \tag{4.4}$$

$$\rho_i = -q_i\left(1 + W_{\mp 1}\left(-\exp\left(\frac{y}{q_i^2} - \frac{z}{q_i} - 1\right)\right)\right), \tag{4.5}$$

To invert the equation we use the formula

$$\rho = z + \frac{B_1}{2}\ln\left(1 + \frac{B_1}{B_2}\rho + \frac{\rho^2}{B_2}\right) - \frac{B_1^2 - 2B_2}{\sqrt{4B_2 - B_1^2}}\mathrm{Arctg}\left(\frac{\sqrt{4B_2 - B_1}\rho}{2B_2 + B_1\rho}\right) \tag{4.10}$$

Then we easily found the iteractive solution

$$a_{it}^{(3)}(\mu) = \left[z + \frac{B_1}{2}\ln\left(1 + \frac{B_1}{B_2}z + \frac{z^2}{B_2}\right) - \frac{B_1^2 - 2B_2}{\sqrt{4B_2 - B_1^2}}\mathrm{Arctg}\left(\frac{\sqrt{4B_2 - B_1}z}{2B_2 + B_1 z}\right)\right]^{-1}. \tag{4.11}$$

The second iteration is easily deduced from this expression, but length to write.

The high-energy solution reads

$$a_{HEP}^{(3)}(\mu) = a_{HEP}^{(2)}(\mu) + \frac{1}{z^3}\left(B_1^2\left(\ln^2\left(\frac{z}{\beta_0}\right) - \ln\left(\frac{z}{\beta_0}\right) - 1\right) + B_2\right). \tag{4.12}$$

In the next table we give comparison of these solutions

$\frac{\mu}{\Lambda}$	$a_s^{(3)}(exact)$	$\Delta a_{it}^{(3)}$	$\Delta a_{it2}^{(3)}$	$\Delta a_{HEP}^{(3)}$
1.5	0.050	-45.3%	-14.5%	-400 %
2	0.037	-24.4%	-6.0%	-91.7 %
4	0.023	-10.0%	-1.65%	-28.5 %
10	0.0161	-5.00%	-0.58%	-16.2 %
100	0.0093	-1.796%	-0.120%	-8.13 %
1000	0.0066	-0.95%	-0.045%	-5.48 %

Here one can see that HEP solution is much worse then iterative, even at very high energy. Double iterative solution is the best. The triple iterative solution improve accuracy by another factor 10.

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where index is -1 for the first solution and $+1$ for the second. Compare it with the NLO solution. Next one should find such y that both solution equal each other. Since $\rho_1 = (\rho_2)^*$, one can restrict the y the the following equation (for $z > 0$)

$$\mathrm{Im}(\rho_1(y)) = 0. \tag{4.6}$$

We cannot find y exactly but iteratively it is given by a rapidly convergent series:

$$\begin{aligned} y &= y_0 + zY + z^2\frac{Y^2}{2B_2^2}\left(B_2 - B_1 Y + Y^2\right) \\ &+ z^3\frac{Y^3}{3B_2^2}\left(1 - \frac{5}{2}\frac{B_1}{B_2}Y + \frac{3}{2}\frac{B_1^2 + 2B_2}{B_2^2}Y^2 - \frac{7}{2}\frac{B_1}{B_2^2}Y^3 + \frac{2}{B_2^2}Y^4\right) + \ldots, \end{aligned} \tag{4.7}$$

where y_0 is the solution of equation

$$\mathrm{Im}W_{-1}\left(-\exp\left(\frac{y_0 - \rho_1^2}{\rho_1^2}\right)\right) = -\frac{\sqrt{4B_2 - B_1^2}}{B_1}\left(1 + \mathrm{Re}W_{-1}\left(-\exp\left(\frac{y_0 - \rho_1^2}{\rho_1^2}\right)\right)\right), \tag{4.8}$$

and

$$Y = \frac{B1}{2}\left(1 + \mathrm{Re}W_{-1}\left(-\exp\left(\frac{y_0 - \rho_1^2}{\rho_1^2}\right)\right)\right)^{-1}.$$

Equation for y_0 can be solved numerically. We found for $N_f = 3$

$$y_0 \simeq -5.1791 \times B_2, \qquad Y \simeq \frac{B_1}{2} \times 0.52386. \tag{4.9}$$

And

$$y \simeq B_2\left(-5.17905 + 0.002604z + 0.00029z^2 + 4 \times 10^{-6}z^3 + 6 \times 10^{-8}z^4\right),$$

so the series is super convergent. **HOWEVER, It is not very practical solution because one should change the sheets of W function with energy.** Much more precise is to invert the equation numerically.

V. RESULT FOR Λ

We have used the normalization point $\mu = M_Z = 91.19$ GeV, where a_s is given by (from MSTW2008nnlo)

$$a_s(M_Z) = 0.00931613. \tag{5.1}$$

Fixing this expression we define Λ_{QCD}. The thresholds are passed as following

$$a_s(N_f, m_f) = a_s(N_f + 1, m_f). \tag{5.2}$$

There are two thresholds to pass $m_b = 4.75$ and $m_c = 1.4$ with $N_f = 5 \to 4$ and $N_f = 4 \to 3$ correspondingly.

We took two solutions double-iterative (that is very close to exact) and HEP (as a standard solution) we obtain the following numbers (in GeV) (superscript is the number-of-loops within solution)

μ	$\Lambda^{(1)}_{it2}$	$\Lambda^{(2)}_{it2}$	$\Lambda^{(3)}_{it2}$	$\Lambda^{(1)}_{HEP}$	$\Lambda^{(2)}_{HEP}$	$\Lambda^{(3)}_{HEP}$
$a_s(M_Z) = 0.00931613$						
$\mu > m_b$	0.083	0.230	0.274	0.083	0.215	0.198
$m_b > \mu > m_c$	0.115	0.324	0.409	0.115	0.313	0.278
$m_c > \mu$	0.138	0.379	0.481	0.138	0.363	0.320
$a_s(1)$	0.028	0.0351	0.378	0.028	0.0397	0.0364
$a_s(1) = 0.0358711$						
$m_c > \mu$	0.213	0.389	0.460	0.213	0.330	0.315

The pre-last line is expression for $a_s(1\text{ GeV})$, that should be compared with the same expression from MSTW2008 package. The last line is the Λ obtained from the normalization at 1 GeV.

One can see that both solutions are rather stable. The HEP solution is a bit more consist with check at 1 GeV, which is probably refers to the fact that it is used in MSTW package.

[1] G. M. Prosperi, M. Raciti and C. Simolo, Prog. Part. Nucl. Phys. **58** (2007) 387 doi:10.1016/j.ppnp.2006.09.001 [hep-ph/0607209].
[2] A. Deur, S. J. Brodsky and G. F. de Teramond, arXiv:1604.08082 [hep-ph].
[3] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, Phys. Rev. Lett. **79** (1997) 2184 doi:10.1103/PhysRevLett.79.2184 [hep-ph/9706430].