Infrared logarithms in Effective Field Theories

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Abstract

The detailed investigation of logarithmical structure of perturbative expansion for Effective Field Theories (EFTs) is presented. It is shown that in massless EFTs the Leading Logarithm (LLog) coefficients satisfy a non-linear recursive equation, which is the consequence of loop-graph topology and the structure of R-operation. The suggested equation is an alternative to the well-known renorm-group equation and an effective instrument for the investigation of perturbative expansion at all orders. Also we show that for theories with $\mathcal{L}_{int} = \mathcal{O}(\phi^4)$ this recursive equation is a consequence of unitarity, analyticity and crossing symmetry. With the help of the suggested methods the leading chiral contribution to parton distributions for pions is obtained and investigated. This contribution plays an important role at small-$x_{\rm Bj}$ and/or large-$b_\perp$ domain of parton distributions.

Abstrakt

Es wird eine detaillierte Untersuchung logarithmischer Strukturen der pertubativen Expansion für effektive Feldtheorien (EFTs) dargestellt. Es wird gezeigt, dass in masselosen EFT die Koeffizienten der führenden Logarithmen (engl. Leading Logarithm) einer nicht linearen rekursiven Gleichung genügen, was eine Konsequenz von loopgraph Topologie und der Struktur der R-Operation ist. Die vorgestellte Gleichung ist eine Alternative zu der bekannten Renormierungsgruppengleichung und ein effektives Instrument zur Untersuchung der perturbativen Expansion in jeder Ordnung. Zuden zeigen wir, dass diese Rekursivgleichung für Theorien mit $\mathcal{L}_{int} = \mathcal{O}(\phi^4)$ eine Konsequenz von Unitarität, Analyzität und Crossingsymmetrie ist. Mit Hilfe der vorgestellten Methoden erhält und untersucht man den führenden chiralen Beitrag zu Partondistributionen für Pionen. Dieser Beitrag spielt eine wichtige Rolle bei kleinem x_{Bj} und/oder der großen $b_\perp$ Domäne der Partondistributionen.

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Contents

Introduction

Effective field theories (EFTs) were introduced ref.[1]. Still there are many unsolved questions related to EFTs. One of these questions is the possibility to compute the higher order terms of the perturbative expansion and the possibility (and also necessity) of their resummation. The present thesis is consecrated on this question, namely, the investigation of the logarithmical terms in the EFT perturbative expansion and the significance of the leading logarithm resummation in some tasks of the particle physics.

The investigation of the perturbative expansion is directly related to the application of the renormalization operation. The renormalization operation consists in the sequential redefinition of the theory constituents. Coupling constants, masses and fields absorb the divergences of the loop integrals that appear during the perturbative calculations. The renormalization procedure divides quantum field theories onto two types – renormalizable and non-renormalizable theories. The distinction between these two types consists in the operator structure of the counterterms (terms which contain the divergences and are subtracted from the bare Lagrangian). For the renormalizable theories the operator structure of the counterterms repeats the structure of the bare Lagrangian, in contrast to the non-renormalizable ones. Therefore, in the non-renormalizable theories one has to introduce new types of interactions and new parameters[1]. This process is uncontrollable in the sense that all new parameters have to be determined experimentally and added to the perturbative expansion. Every next order of the calculation gives more additional parameters, and so on.

The EFTs are, usually, non-renormalizable theories. But in fact a proper EFT derived from the fundamental theory (such as quantum chromodynamic (QCD)) by integrating over the appropriate degrees of freedom is already finite, because all divergences were taken into account in the "parent" theory. Therefore, one can make the calculation and the renormalization in the EFT in the same manner as in a usual renormalizable theory.

In the particle physics the most popular EFTs are theories that describe the low energy interaction of hadrons. The derivation of such EFTs from the QCD is a very difficult task, which has never been done (except the lowest order terms, see e.g.

[1]We tractate the theory, which needs the addition of finite number of terms, as renormalizable one, although some authors define it as non-renormalizable.

[2],[3]). Therefore, one has to construct an EFT Lagrangian starting from the general idea of hadron interaction and to determine the low energy constants experimentally. The general method for the construction of an EFT was suggested in the seventies [1]. The main idea was to employ the general properties of the physical quantities, such as analyticity and symmetries. As it is shown in [1], "*the most general possible S-matrix consistent with perturbative unitarity, analyticity, cluster decomposition and assumed symmetry principles*" follows from "*the most general possible Lagrangian with assumed symmetry principles*".

The most general possible Lagrangian contains an infinite number of operators and every operator comes with its own coupling constant. One has to introduce some hierarchy of the couplings in order to perform the systematical perturbation expansion. The natural choice of the expansion parameter is E/Λ, where E is the characteristic energy of the process, and Λ is the lowest dimension coupling constant (note that in the non-renormalizable field theory the coupling constants are dimensional). Obviously, the higher dimension terms of the Lagrangian contribute to the higher order of the momentum expansion. Therefore, the calculation of the given order of the momentum expansion requires a finite number of the Lagrangian terms. The scheme of the selection of the Lagrangian terms and diagrams, which contribute to the given order of the momentum expansion, is called Weinberg counting scheme [1].

The number of the parameters in the Lagrangian increases from order to order. As an example, we take the chiral perturbation theory (ChPT). The ChPT is one of the most popular, investigated and powerful EFT nowadays (for a modern review see [5],[6]), and we will often appeal to it in the text. ChPT describes the dynamics of the light bosons (pions, kaons, eta-meson), as a dynamic of the Goldstone bosons of the spontaneously broken chiral symmetry. The leading order Lagrangian of ChPT contains one coupling [7], [8], namely $F_\pi \approx 93$ MeV — the pion decay constant. The next-to-leading order of the Lagrangian (and hence the next-to-leading order of the momentum expansion) contains 10 parameters. This order of the chiral expansion allows one to make predictions for form factors, amplitudes, and other quantities up to $E \sim 300 - 400$ MeV within several percent accuracy, e.g. [9],[10]. But the next-to-next-to-leading order Lagrangian contains already 90 parameters [11]. And one can not find enough independent observables to fix all parameters.

The usage of the Weinberg counting scheme gives good enough results for the low energy behavior of amplitudes, but the orderliness of the Weinber scheme can be broken in the presence of another, except the energy E, scale. An example of such a situation is the series of the chiral corrections to the partonic distribution functions. In this case an additional scale parameter – the light cone distance – appears. The additional scale parameter spoils the usual momentum expansion counting scheme since the momentum order of some terms can be compensated by the light cone distance. Therefore, one has to perform a detailed investigation of the perturbative series in order to extract the leading terms. Often, one needs to sum up some part of the whole perturbative series in order to obtain the leading behavior.

It is well-known that the perturbative expansion in the renormalizable theory can be partially resumed. The summation is going over the powers of logarithms, which appear at every order of the expansion. These logarithmical contributions are strongly related to each other and to the renormalization procedure. Using the renormalization group (RG) technique one can obtain the logarithmical series without a direct calculation of every term. In particular, the leading logarithmical contribution can be obtained with only an one-loop calculation, the next-to-leading logarithmical contribution can be obtained with a two-loop calculation, and so on.

The logarithmical expansion in an EFT also makes sense, since the number of the parameters at a given order of the logarithmical expansion is the same as in the same order of the momentum expansion. For example, the leading logarithmical approximation depends on the parameters of the lowest order Lagrangian only, although it contains all orders of the momentum expansion. In the contrast to renormalizable theories, the logarithmical terms can not compete with the regular power terms of the usual perturbative expansion even in the deep infra-red (IR) region. However, in the case of the broken order of perturbative expansion they can be very important.

The usual methods of RG, which were used in the renormalizable theories, do not apply to the case of the non-renormalizable theories without modifications. The first application of RG in the EFT was done by Weinberg [4], who calculated the double logarithm coefficient of the $\pi\pi$ scattering amplitude using one-loop calculations and renorminvariance principle. The RG principles at a finite order of expansion were used in many papers in order to check the calculations and for the predictions of double logarithm structures, e.g. [12],[13]. The complete formulation of the application of RG to EFTs at any order was developed by Buchler and Colangelo [14]. As one of the results, they showed that the leading logarithm behavior of the perturbative expansion is given by the one-loop renormalization of the complete Lagrangian. However, the problem of the calculation of the one-loop counterterms to all order Lagrangian is very difficult since it contains an infinite number of terms. In the recent papers by Bijnens and Carloni [15],[16] the five-loop leading logarithm coefficients for the different objects in the massive $O(N)$ σ-model were found using the technique described in ref. [14].

There were many attempts to use other ways to describe the leading logarithm behavior of non-renormalizable theories. Here we list some of them. One of the popular ideas is to build formally renormalizable constructions in EFT and with their help to find the required behavior, e.g. [17], [18]. The leading logarithmical structure can be simplified using some specifics of Lagrangian symmetries, e.g. Lagrangians with $O(N)$ symmetry at large-N [19],[20], supersymmetrical Lagrangians [21]. One can use the dispersion relations and crossing symmetry [22]. Practically, none of these methods is universal or can be used at higher orders.

The main aim of this thesis is to derive the method for obtaining the leading logarithm behavior of amplitudes in a massless EFT. The massless EFTs have simpler properties than the massive ones; nevertheless, they have all main features of an EFT and often appear in the applications. Particulary, we investigate the massless EFTs with the lowest order interaction term build up of four fields. Since these types of EFTs are the most frequently used, we will call them as EFTs with ϕ^4-type of interaction.

The second aim of the thesis is to apply the leading logarithm summation to the investigation of the meson partonic functions, and solve the problem of the breaking down of their chiral expansion [23]. The singular contributions, which appear during the evaluation of the parton distributions in ChPT, prohibit the usage of the finite order of the chiral expansion and have to be resummed. The singular terms dominate in the region of low x, and their resumed expression gives the important information about the hadron structure, such as asymptotic behavior in the impact parameter space, relative contribution of the pion cloud to the hadron.

In chapter I, we review the structure of the perturbative expansion in the EFT and the Weinberg counting scheme. We discuss the structure of the R-operation in the non-renormalizable theories and its connection with the new operators appearing in the theory. We also review the basics of the dimension regularization and the renorminvariance principle. Finally, we review the main statements of the RG approach to EFT [14].

Chapter II is devoted to the method of obtaining the leading logarithm to all orders based on the renorminvariance principle. First, the properties of the massless EFTs are discussed. In the massless EFT some hierarchy of the graphs with respect to their contribution to the logarithm structure is present. This hierarchy simplifies the picture dramatically and together with the renorminvariance allows one to write the non-linear recursive equation for the leading logarithm coefficients of the 4-point amplitude. The kernel of the equation is the one-loop beta-function, which has to be calculated for all order Lagrangian. We build the all-order effective Lagrangian for the case of the $O(N+1)/O(N)$ σ-model and calculate the required beta-function. After this we investigate the equation from both numerical and analytical points of view. The obtained coefficients of the 4-point amplitude are used as parameters in the recursive equations on the leading logarithm behavior of other quantities such as form factors and 6-point amplitudes, which are also considered in this chapter. Also, we present the set of necessary expressions (equations, beta-functions) for $SU(N)\times SU(N)$ σ-model and matrix model. The main results of the second chapter are published in refs. [24] and [25]. The presented method is quite general and can be used for any massless theory regardless to its interaction structure.

In chapter III, the alternative approach of obtaining the leading logarithm coefficients using the unitarity and crossing symmetry is described. Results of this approach are equivalent to the results obtained in chapter II. The form of the equations and the calculation of their kernel are easier than in the direct loop-calculation method, but the range of applicability of this dispersion approach is narrower. For example, it can not be applied to the theories with cubic interaction or to the Green functions. Using this method we obtain the recursive equations for the 4-point amplitude and the form factor in the theory with an arbitrary global symmetry. We find that the one-loop beta-functions in the theory with leading ϕ^4 interaction has the universal form in terms of crossing matrices. This result connects the concepts of the renormalization group and the analytical properties of the amplitude. The expression for the beta-functions is also valid for the renormalizable theories. The calculation of the crossing matrices is much simpler than the calculation of the loop integrals. As in chapter II, we illustrate the application of the method considering $O(N+1)/O(N)$ and $SU(N)\times SU(N)$ σ-models. Also, we present the generalization of the equation for the case of an arbitrary D-dimension and for the case of mixed renormalizable-non-renormalizable interactions. The main results of this chapter are published in [26].

In chapter IV, we apply the methods obtained in the previous chapters to the calculation of the singular chiral corrections to the pionic parton distributions (PDFs and GPDs). We remind the reader the basics of the application of EFT to the deep inelastic processes and explain the source of the singular contributions. After this we make the calculation of the required terms and perform their summation. Finally, we discuss the properties of the obtained results and their influence on our understanding of the hadron structure. The main results of this chapter are published in [27] and [28].

I

Basics of effective field theories

In the present chapter we review the basics of EFTs and introduce necessary notations and definitions. We start with the description of the general EFT Lagrangian with scalar fields and the Weinberg counting rule and then discuss the structure of the perturbation expansion in EFTs. We also discuss the structure of the renormalization procedure and its influence on the structure of the perturbation series. After that we discuss the renormalization group equations (RGE) in EFTs and their connection with the infrared logarithms of the theory.

I.1 Structure of the perturbation expansion in EFT

The Lagrangian of an EFT contains all possible operators with the assumed symmetry. It is impossible to take into consideration all possible operators simultaneously, because for that one has to determine all parameters of the Lagrangian. In order to perform the self-consistent perturbative expansion one has to fix the small parameter of the expansion. In the Weinberg counting scheme [1], the external kinematical variables play the role of small parameters. It is assumed that all external momenta and masses of particles are much less then the scale of EFT breaking, $p_i, m_i \ll \Lambda_{EFT}$. Therefore, the Lagrangian can be represented as the following sum

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + ... \,, \tag{I.1-1}$$

where each $\mathcal{L}_n$ contains only the terms with the n-th power of momenta variables, i.e. ∂^n, m^n etc. Note, that in exp. (I.1-1) we do not write the odd-index terms, because in the thesis we discuss only EFTs with boson fields. The terms with odd power of momenta are possible, but they accompany fermion fields.

Every order of the Lagrangian can contain an infinite number of the terms with a different number of the fields[1]:

$$\mathcal{L} = \underbrace{-\frac{1}{2}\phi(\partial^2 - m^2)\phi + g_1^{(4)}\phi^2\partial^2\phi^2 + g_1^{(6)}\phi^4\partial^2\phi^2 + \ldots}_{\mathcal{L}_2} + \underbrace{g_2^{(2)}\phi\partial^4\phi + g_{2,0}^{(4)}\phi^2\partial^4\phi^2 + g_{2,2}^{(4)}(\phi\partial_\mu\partial_\nu\phi)^2 + \ldots}_{\mathcal{L}_4} + \ldots\ , \tag{I.1-2}$$

where the coefficients g are coupling constants of operators. Through the whole text we use the universal notation for coupling constants. We denote the coupling constant of the operator with k fields from $\mathcal{L}_{2n}$ by $g_n^{(k)}$. In order to distinguish between different terms with k fields in $\mathcal{L}_{2n}$, we add an auxiliary lower index to the couplings, e.g. the last two term of $\mathcal{L}_4$ in exp. (I.1-2). Often, we omit the auxiliary index in order to concentrate on the general properties of an EFT.

There are two important remarks on the EFT Lagrangians in the form (I.1-2). First, the coupling constants of the same order Lagrangian are not independent. The symmetries of the theory imply the relations between them (see the example of ChPT below). But since we will mainly consider the topology of graphs, it is more convenient to denote the coupling for every term separately. Second, the propagator of the particle is given by $\mathcal{L}_2$. 2-field terms of the higher order Lagrangians should not be added to the propagator. They form the set of 2-vertices, which have to be taken into account as usual vertices. This approach is unusual for the renormalizable theories and is related to the structure of the renormalization, as it will be discussed below.

The canonical momentum dimension of the coupling $g_n^{(k)}$ is

$$\left[g_n^{(k)}\right] = D - k\frac{D-2}{2} - 2n, \tag{I.1-3}$$

where D is the number of space-time dimensions and $\frac{D-2}{2}$ is the dimension of the field ϕ. Therefore, at $D \geqslant 4$ all couplings have a negative momentum dimension. The only exception is $g_0^{(4)}$ at $D = 4$, this case satisfies the usual renormalizable ϕ^4-theory.

Let us consider the matrix element of a process involving N_ϕ external particles ϕ. Its momentum dimension is

$$[\mathcal{M}] = D - N_\phi\frac{D-2}{2}. \tag{I.1-4}$$

The lowest order term of the matrix element momentum expansion is given by the tree diagrams with the interaction vertices from $\mathcal{L}_2$. The higher order of the expan-

[1] In the thesis we consider only Lagrangians with even number of fields. Some statements are correct only in this class of EFTs. In particular, this allows us to use compact notations, which is not possible in the theory with *any* number of fields. However, many results obtained for these Lagrangians can be generalized on a wider class of Lagrangians, see chapter II section 7.

sion is given by the loop- and tree- graphs including couplings from the higher order Lagrangians.

The diagram which contributes to the matrix element (I.1-4), has the following momentum order, or the power of energy scale,

$$[E] = \sum_p pN_p - 2N_i + DN_l,$$

where N_p is the number of vertices with p derivatives, N_i is the number of the internal propagators and N_l is the number of the loops in the graph. Taking into account the dependence between the number of loops, lines and vertices in an arbitrary graph:

$$N_l = N_i - \sum_p N_p + 1,$$

one obtains the power of the energy scale of the diagram in the form

$$[E] = \sum_p (p-2)N_p + (D-2)N_l + 2. \tag{I.1-5}$$

The momentum dimension of the diagram is compensated to the dimension of the matrix element (I.1-4) by $N_v = \sum_p N_p$ dimensional constants (I.1-3) of the Lagrangian (I.1-2). The overall dimension of the couplings in the graph is

$$[\sum g] = [\mathcal{M}] - [E] = \frac{D-2}{2}\big(2 - 2N_l - N_\phi\big) - \sum_k (k-2)N_k. \tag{I.1-6}$$

Therefore, the number of diagrams which contribute to the given order of the momentum expansion, is finite. These diagrams contain only the coupling constants with $[g] \leqslant [\sum g]$. It is important to note one exceptional case: if a Lagrangian contains the dimensionless coupling, i.e. $g^{(4)}_{0(D=4)}$, this coupling can contribute an arbitrary number of times to the given order of $[E]$.

Consequently, the momentum expansion of the matrix element has the form

$$\mathcal{M} = \sum_{n=0}^{\infty} g^{N_v} E^n f_n(E, p_i, m_i), \tag{I.1-7}$$

where g^{N_v} is the product of different couplings with the overall momentum dimension (I.1-6), E is the dimensional expansion parameter, f is a dimensionless function of its arguments.

The ultraviolet (UV) regularization and renormalization of loop-diagrams add the dependence on the renormalization point μ into the function f. Functionally, the parameter μ appears only as the argument of logarithms. This is the consequence of the counterterms locality, see e.g. [29], [30]. Generically, a diagram with N_l loops can produce $\ln(\mu)$ in different powers, but not higher than the number of loops N_l. Therefore, one can rewrite exp. (I.1-7) in the form

$$\mathcal{M} = \sum_{n=0}^{\infty} E^n \sum_{m=0}^{N_{l\ max}(n)} g^{N_v} \ln^m \left(\frac{\mu}{E}\right) f_{n,m}(E, p_i, m_i), \tag{I.1-8}$$

where $N_{l\ max}(n)$ is the maximum number of loops in diagrams of E^n order and f is a dimensionless function of its variables.

At the given order of the momentum expansion, the number of loops can not exceed the number dictated by eqn. (I.1-5). The maximum number of loops is reached when all vertices in the diagram are of the lowest dimension, i.e. from $\mathcal{L}_2$ (if $\mathcal{L}_2$ contains interaction terms). Thus, at the given order of the momentum expansion the maximum power of $\ln(\mu)$ is multiplied only by the constants from the lowest order Lagrangian:

$$\mathcal{M} \ = \ \sum_{n=0}^{\infty} E^n \Big[g_1^{N_v} \ln^{N_{l\ max}-1} \left(\frac{\mu}{E}\right) f_{n,N_{l\ max}}(E, p_i, m_i) + ... \Big]. \tag{I.1-9}$$

These terms of the matrix element will be called the leading logarithms (LLogs), i.e. the terms with the maximum power of logarithms at the given order of perturbative expansion. The next-to-leading logarithms (NLLog) are the terms with the next-to-leading power in the expansion term. The dependence of the LLog terms only on the coupling constants of $\mathcal{L}_2$ means that in principle one can obtain the LLog terms at all orders of momentum using only $\mathcal{L}_2$ part of the Lagrangian (I.1-2).

I.2 The renormalization

The renormalization procedure consists in the successive subtraction of UV divergences from the Lagrangian parameters. The order of operations is the following. First of all, one introduces the regularization and explicitly extracts UV divergences from the loop-integrals. Secondly, one redefines the couplings constants, masses, and fields in such a way that all divergences at the given order of the perturbative expansion disappear. Thirdly, one takes off the regularization and obtains the finite expression. The details of the renormalization procedure can be found in many text-books, e.g. [30],[32]. In this section we discuss only several points needed for the future explanations and the features of the renormalization of EFTs.

In the thesis we are going to use the dimensional regularization, which is the most simple and investigated method of the regularization of the UV divergences in the loop calculation. The main idea of the dimensional regularization consists in the analytical continuation of the multi-dimensional gaussian integral, which is the basis of the loop-calculation, to the D-dimension space, where D is a non-integer number.

Therefore, the singularities of Feynman diagrams in the dimensional regularization are represented by inverse powers of ε. After the R-operation, D is turned to its physical value. We use the usual definition $D = D_0 - 2\varepsilon$, where D_0 is the canonical dimension of space.

At the same time the momentum dimension of the couplings in the dimensional regularization is kept fixed. In order to have the correct dimension of the Lagrangian one introduces the scale μ^2 in such a power that compensates the change of the space-time dimension. The momentum dimension of the coupling constant is given by (I.1-3) with $D = D_0$. The dimension of the operator with k fields from the $2n$-th order of the Lagrangian is

$$[\mathcal{L}_{2n}^{(k)}] = k[\phi] + 2n + [g_n^{(k)}] + [\text{Compensation}] = D.$$

Therefore, in the dimensional regularization every coupling constant is transformed as

$$g_n^{(k)} \xrightarrow{\text{dim.reg.}} \mu^{(D_0-D)\frac{k-2}{2}} g_n^{(k)} = \mu^{\varepsilon(k-2)} g_n^{(k)}. \tag{I.2-10}$$

Other types of regularizations, e.g. cut or Pauli-Villars, introduce the dimensional parameter as a regulator, and they do not need any additional scale.

The substraction procedure consists in the selection of such additional terms to the Lagrangian parameters, called counterterms, that the additional diagrams generated by the added terms cancel the divergences. This redefinition of parameters is universal in the sense that the redefinition of the finite set of parameters make any matrix element finite. Practically, one usually renormalizes diagrams "in the air" canceling the divergences by the recursive method and calculating the counterterms to the coupling separately.

The main difference of the renormalizable and non-renormalizable theories is in the structure of the counterterm Lagrangian. In the renormalizable theory the loop correction to the Lagrangian term has the same operator structure as the term itself

$$\mathcal{L}(g_0) \sim \mathcal{L}_{ct}(g_0),$$

where sign $\sim$ means the operator equivalence. In the pure non-renormalizable theory the loop-correction to the Lagrangian term produces the operator structure of the higher order Lagrangians. Therefore, the counterterm Lagrangian of n-th order contains the parameters of the lower order Lagrangians

$$\mathcal{L}_{2n}(g_n) \sim \mathcal{L}_{2n,ct}(g_1, g_2, ..., g_{n-1}).$$

Some technical details of the renormalization procedure are presented in the next section.

Note that no *new* terms are added to the Lagrangian (I.1-1) since we assume that EFT contains *all* possible terms by definition. This is the "psychological" difference between "just" non-renormalizable theories and EFTs. However, the construction of the counterterm Lagrangian can be used as a way to find the higher order Lagrangians. It is also possible that the counterterm Lagrangian does not contain all terms of the Lagrangian at the given order.

The renormalized coupling constants depend on the parameter μ, which was introduced during the regularization. This dependence remains after the taking off the regularization. Therefore, the result of the calculation at the finite order of the perturbative expansion depends on the regularization procedure. However, the expression for the complete perturbation expansion, which contains an infinite number of terms, does not depend on μ. This property is called the renormalization group invariance (RG invariance). The RG invariance can be proven for the renormalizable theories, e.g. [29], and has not been proven for non-renormalizable ones. But since one assumes that EFT is the consequence of some fundamental renormalizable QFT, one can also assume that EFT is RG invariant.

I.3 Chiral Perturbation theory

ChPT is the most popular EFT for the description of the low energy hadron interaction. It is based on the idea that the chiral symmetry, which is approximately presented in the QCD Lagrangian, $SU_L(3) \times SU_R(3)$ is spontaneously broken to the $SU(3)_V$ [33]. The Goldstone bosons related to the spontaneously broken $SU(3)$ symmetry are pions, kaons and eta-meson. Their non-zero mass appears due to the small explicit breaking of the chiral symmetry by the presence of the quark masses. Therefore, it is assumed that by integrating the quark and gluon degrees of freedom of the QCD partition function one can represent the partition function as

$$Z(v,a,s,p) = \int D\bar{q}\, Dq\, DA e^{i\int dx \mathcal{L}_{QCD}(\bar{q},q,A)} = \int DU e^{i\int dx \mathcal{L}_{ChPT}(U)},$$

where v (a,s,p) are vector (axial, scalar, pseudo-scalar) currents, and U is the field of Goldstone bosons.

The Lagrangian of ChPT was constructed in [7] and [34] for the cases of $SU(2)$ and $SU(3)$ symmetries respectively. The lowest order Lagrangian has the form

$$\mathcal{L}_2 = \frac{F^2}{2}\mathrm{tr}\Big[\partial_\mu U^\dagger \partial^\mu U + m^2(U + U^\dagger)\Big], \tag{I.3-11}$$

where U is the unitary matrix constructed from the Goldstone fields. In exp. (I.3-11) we have omitted the current terms and consider the $SU(2)$ ChPT version for simplicity. The usual representation for the matrix U is $U = \exp[i\pi^a\sigma^a]$ where σ^a

is a generator of the $SU(2)$ group and π^a is the pion field. It is the only possible Lagrangian with $SU(2)_V$ symmetry of dimension 2.

The coupling constant F is the pion decay constant $F_\pi \simeq 93$ MeV. Thus, one can estimate a validation window of the theory as $E^2 \ll (4\pi F_\pi)^2 \simeq 1.36$ GeV^2, where the factor $(4\pi)^2$ is the usual perturbative multiplicator of the expansion parameter. However, this value is not correct, because a new physical effects arise much earlier on the energy scale. The realistic upper energy boundary for ChPT application is $E \sim 300$ MeV^2.

The tree order calculation with the Lagrangian (I.3-11) gives the first term of the momentum expression $\sim E^2$. The correction to this term is given by one-loop diagrams with vertices from $\mathcal{L}_2$ (I.3-11). According to the momentum counting (I.1-5) these diagrams produce the term of order E^4. Therefore, the divergences of these diagrams have to be compensated by the counterterms of $\mathcal{L}_4$. The counterterm Lagrangian has the form [34]

$$\begin{aligned}\mathcal{L}_4^{\mathrm{ct}} &= \frac{1}{(4\pi)^2}\frac{1}{\varepsilon}\Big[\frac{1}{3}\mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U)^2 + \frac{2}{3}\mathrm{tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{tr}(\partial^\mu U^\dagger \partial^\nu U) \\ &\qquad + \frac{3}{2}m^4 \mathrm{tr}(U)^2 + 2m^2 \mathrm{tr}(U)\mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U)\Big],\end{aligned} \tag{I.3-12}$$

where ε is a parameter of the dimensional regularization.

The symmetry consideration gives the next order Lagrangian in the form [34]

$$\begin{aligned}\mathcal{L}_4 &= \frac{l_1}{4}\mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U)^2 + \frac{l_2}{4}\mathrm{tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{tr}(\partial^\mu U^\dagger \partial^\nu U) \\ &\qquad + \frac{l_3}{4}m^4 \mathrm{tr}(U)^2 - \frac{l_4}{2}m^2 \mathrm{tr}(U)\mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U)\Big].\end{aligned} \tag{I.3-13}$$

One can see that the operator contents of Lagrangians (I.3-12) and (I.3-13) coincide. Thus, the renormalized constants of $\mathcal{L}_4$ have the form

$$l_i = l_i^r + \gamma_i \lambda, \quad \gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{I.3-14}$$

where

$$\lambda = \frac{\mu^{-2\varepsilon}}{(4\pi)^2}\Big(\frac{1}{\varepsilon} + \text{Scheme constant}\Big).$$

Note, that here we present only a part of $\mathcal{L}_4$. The complete Lagrangian (I.3-13) contains 10 operator structures and, hence, 10 different coupling constants. Moreover, some of the operators have no counterterms, i.e. they are not presented in $\mathcal{L}_4^{\mathrm{ct}}$ and can be restored only by the symmetry consideration.

The loop calculation with the ChPT Lagrangian takes place in terms of particle fields, i.e. π^a. Expanding the matrix U around $\pi^a = 0$, $U = 1 + i\frac{\lambda_a \pi^a}{F} - \frac{1}{2}\frac{(\lambda_a \pi^a)^2}{F^2} + \dots$,

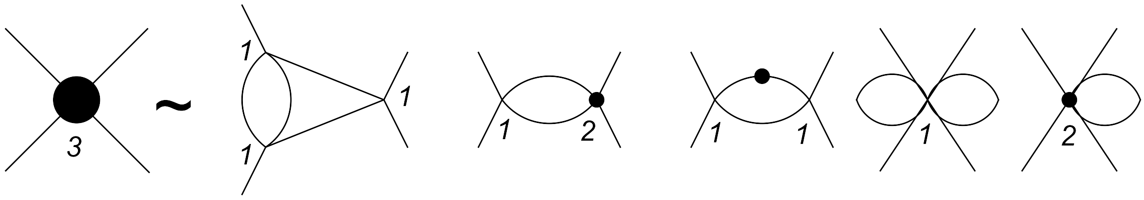

Figure I-1: The examples of graphs contributing to the renormalization of $g_3^{(4)}$. The numbers correspond to the index of the corresponding vertex constant.

one obtains the Lagrangians (I.3-11), (I.3-13) in the form (I.1-2):

$$\mathcal{L}_2 = -\frac{1}{2}\pi^a(\partial^2 + m^2)\pi^a - \frac{1}{8F^2}\pi^2(\partial^2 + m^2)\pi^2 + \mathcal{O}(\pi^6) \tag{I.3-15}$$

$$\begin{aligned}\mathcal{L}_4 \;=\; & -\frac{1}{2}\frac{m^2}{F^2}(l_4\pi^a\partial^2\pi^a + l_3\pi^2) + \frac{l_1}{F^4}\left(\partial_\mu\pi^a\partial_\mu\pi^a\right)^2 + \frac{l_2}{F^4}\left(\partial_\mu\pi^a\partial_\nu\pi^a\right)^2 \\ & + \frac{l_3m^4}{2F^4}\left(\pi^2\right)^2 + \frac{l_4m^2}{F^4}\left(\pi^2\partial_\mu\pi^a\partial_\mu\pi^a - \frac{1}{2}\partial_\mu\pi^a\partial_\mu\pi^2\right) + \mathcal{O}(\pi^6)\end{aligned} \tag{I.3-16}$$

From the comparison of these two expressions with exp. (I.1-2) one sees that in our notation the coupling $\frac{1}{F^2} = g_1^{(4)}$, $\frac{l_{1(2)}}{F^4} = g_{2,1(2)}^{(4)}$ and so on. The higher field coupling constants $g_{1(2)}^{(k>4)}$ are expressed only thorough the $g_{1(2)}^{(4)}$ due to the chiral symmetry.

The two-loop contribution of $\mathcal{L}_2$ diverges quadratically and has the momentum dimension 6, (I.1-5). At the same chiral order one has one-loop diagrams, which are composed of one vertex from $\mathcal{L}_4$ and one vertex from $\mathcal{L}_2$. These two classes of diagrams have the same counterterm operator structure. For example, schematically, the renormalization of the coupling $g_3^{(4)}$ has the form

$$\begin{aligned}g_3^{(4)} \;\sim\; & g_3^{(4)r} + (g_1^{(4)})^3\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + g_1^{(4)}g_2^{(4)}\frac{1}{\varepsilon} + (g_1^{(4)})^2 g_2^{(2)}\frac{1}{\varepsilon} \\ & + g_1^{(8)}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + g_2^{(6)}\frac{1}{\varepsilon}.\end{aligned} \tag{I.3-17}$$

Examples of the graphs, which correspond to each term of eqn. (I.3-17), are shown in fig.I.1. The one-loop diagrams with two constants from $\mathcal{L}_4$ have dimension 8 and such diagrams can be renormalized using $\mathcal{L}_8$, and so on. Note, that due to the global Lagrangian symmetries $g_1^{(8)} = (g_1^{(4)})^3$ and $g_2^{(6)} = g_1^{(4)}g_2^{(4)}$.

ChPT has many extensions, e.g. baryons chiral perturbation theory [35], [36], ChPT with vector particles [37], unitarized ChPT [38]. There are also a lot of detailed reviews on different aspects of ChPT, e.g [5],[39], [40].

I.4 Renormalization group equations

The RG invariance is the property of the complete perturbative expansion. Consideration of this property allows one to fix some terms of the perturbative series at all orders. This can be done with the help of the renormalization group equations (RGEs). The RGEs are the standard tool for the renormalizable theories, see e.g. [32]. In the non-renormalizable theories the application of the RGEs is much more difficult. The first application of RGE in an EFT was made by Weinberg [1], who found the LLog coefficient at p^6-order of the $\pi\pi$-scattering amplitude using only the one-loop calculations. Since that time, large research effort was devoted to RGE in the non-renormalizable EFTs. The first complete method of building RGE based on the RG invariance principle was proposed in ref [14]. The main idea is to demand the RG invariance not for the whole Lagrangian, but separately for every operator structure at every order of the quantum correction. This consideration leads to the infinite set of deferential equations on the run of the couplings.

The bare Lagrangian of the n-th order contains the renormalized Lagrangian and its counterterm Lagrangian. Both structures can be expanded in the minimal basis of independent operators $\mathcal{O}^{(k)}_{ni}$:

$$\mathcal{L}_n = \sum_{k=2}^{\infty} \mu^{\varepsilon(k-2)} \sum_{C=0}^{M_n} g^{(k)}_{nC} \mathcal{O}^{(k)}_{nC}, \tag{I.4-18}$$

where M_n is the number of the operators in the basis at the given order. The main difference between the renormalizable and non-renormalizable theories in such approach[2] is the n-behavior of M_n. M_n is an increasing function of n for non-renormalizable theories and constant for renormalizable ones.

The RGE follows from the requirement that every bare Lagrangian (I.4-18) is RG invariant, hence

$$\begin{aligned} 0 &= \mu^2 \frac{d\mathcal{L}_n}{d\mu^2} \qquad\qquad \text{(I.4-19)} \\ &= \sum_k \mu^{\varepsilon(k-2)} \left\{ \varepsilon(k-2)\Big(\mathcal{L}^{(k)}_n + \sum_{j=1}^{n} \frac{A^{(k)}_{nj}}{\varepsilon^j}\Big) + \mu^2 \frac{d\mathcal{L}^{(k)}_n}{d\mu^2} + \sum_{j=1}^{n} \mu^2 \frac{d}{d\mu^2} \frac{A^{(k)}_{nj}}{\varepsilon^j} \right\}. \end{aligned}$$

[2] To be precise one should consider the quantum corrections of the Lagrangian around the solution

$$\mathcal{L} = \sum_{n=0}^{\infty} \hbar^n \mathcal{L}_n.$$

However, this specification is needed only for the renormalizable theories, since for the pure non-renormalizable theories the quantum expansion practically coincides with the momentum expansion (I.1-1).

The μ-dependance of the Lagrangian is described by its β-functions

$$\mu^2 \frac{d\mathcal{L}_n^{(k)}}{d\mu^2} = \mathcal{B}_n^{(k)} - \varepsilon \frac{(k-2)}{2} \mathcal{L}_n^{(k)}. \tag{I.4-20}$$

One of the main statements of the R-operation for the renormalizable theories is that $\mathcal{B}$ contains no singularities. Therefore, we demand the absence of singularities for the EFT β-functions also. By definition, $\mathcal{B}$ consists of the same operators as the Lagrangian (I.4-18)

$$\mathcal{B}_n^{(k)} = \sum_{C=0}^{M_n} \beta_{nC}^{(k)} \mathcal{O}_{nC}^{(k)}, \tag{I.4-21}$$

where β denotes beta-functions of the corresponding couplings. Thus, eqn. (I.4-18) and eqn. (I.4-20) give together

$$\mu^2 \frac{dg_{nC}^{(k)}}{d\mu^2} = \beta_{nC}^{(k)} - \varepsilon \frac{(k-2)}{2} g_{nC}^{(k)}. \tag{I.4-22}$$

This is RGE for the couplings. In the non-renormalizable theories it has exactly the same form as in the renormalizable theories, the only difference is that in the non-renormalizable theories the number of RGEs is finite. Note, that since in our notation the most part of g's are related to each other, their β functions are also not independent.

The Green function G calculated from the RG invariant Lagrangian is also RG invariant, which leads to the following relation

$$0 = \mu^2 \frac{d}{d\mu^2} G = \mu^2 \frac{\partial G}{\partial \mu^2} - \hat{H} G, \tag{I.4-23}$$

where the operator $\hat{H}$ is

$$\hat{H} = -\sum_{n,k,C} \beta_{nC}^{(k)} \frac{\partial}{\partial g_{nC}^{(k)}}, \tag{I.4-24}$$

where the minus sign is put for the future convince. The formal solution of eqn. (I.4-23) is

$$G(\mu^2, g(\mu^2)) = \exp\left[\ln\left(\frac{\mu^2}{\mu_0^2}\right)\hat{H}\right] G(\mu_0^2, g(\mu_0^2)), \tag{I.4-25}$$

where μ_0 is the normalization point. At the normalization point the couplings are defined exactly, i.e. their quantum corrections are zero and one can use their experimental values. The Green function at this point is given by the sum of all possible tree diagrams. It is very important for the self-consistence that all couplings must be defined at the same scale μ_0.

Eqn. (I.4-23) is an analog of the Callan-Symanzik equation for the renormalizable theories [41]. In contrast to the Callan-Symanzik equation, eqn. (I.4-23) contains no anomalous dimensions of fields, since we did not introduce the renormalization of the field itself. The field renormalization procedure is replaced by introducing the infinite set of the two-field operators with the dimensional constants, $g^{(2)}$. The μ-evolution of $g^{(2)}$ constants is equivalent to the presence of the anomalous dimension in the usual form of the Callan-Symanzik equation. Thus, the tree diagrams with 2-vertices have to be presented in the Green function at the normalization point μ_0 on the same ground as others tree diagrams.

In the original paper [14] the authors describe several properties of the operator $\hat{H}$. In the next chapter we obtain these properties using another technique and notations.

Accounting of the RG invariance (I.4-25) corresponds to the all order resummation of the logarithmic contributions. It turns out that the LLogs in any field theory can be obtained only through the one-loop calculation, the NLLogs can be obtained through the two-loop calculation and so on. Under the term "one-loop calculation" we understand the calculation of the one-loop counterterm to the whole Lagrangian with an infinite number of terms. On the other hand, LLogs appear only in the diagrams with initial Lagrangian vertices, see eqn. (I.1-9). This does not mean that the knowledge of the lowest order Lagrangian is enough for the determination of the complete one-loop counterterm. The lowest order Lagrangian gives only the information on the highest pole counterterm. The calculation of the one-loop counterterm to the whole Lagrangian seems an absolutely impossible task, just as the calculation of the n-loop diagram in a non-renormalizable theory. However, it can be done in a wide class of theories, as it will be shown in the next chapter.

Although the calculations of the μ-dependence and, particulary, LLog behavior seems to be very similar in EFTs and in fundamental theories, the physical meaning of it is completely different. In the fundamental theories the μ-dependentce defines the run of the amplitudes in the whole region of the perturbation expansion. In principle, the running of the coupling defines the region of the expansion by itself. The well-known example is the asymptotical freedom, which is a consequence of RGE and it says that the perturbative expansion is valid in the region of large energies. The coupling in the renormalizable theories always runs as $\sim \ln^{-1}\mu^2$, therefore, the LLogs dominate in the perturbative expansion, and the expansion over them is well-founded. In an EFT the perturbative expansion is valid only in the narrow region. The run of the couplings is very complex, usually, they run as a positive power of logarithms. So, the extraction of one or another term from the perturbation series has to be validated separately for every case. The example of the extraction of the dominant terms connected with the external operator structure and the kinematic specifics, will be shown in chapter IV. The only selected case is a massless EFT. In the absence of masses the expansion over the LLog terms gives the IR asymptotic behavior of the given order of the expansion. The absence of other dimensional parameters except momenta and μ^2 allows one to identify the renormalization group logarithms as the IR logarithms.

II

Leading Logarithm in massless EFT

A massless EFT is the simplest case for the analysis of the logarithm structure. The absence of masses in the theory grants many simplifications. Some of these simplifications allow one to find the relations between the LLog coefficients at all orders. This relations has a form of recursive equations. First time the equation was obtained for the $\pi\pi$ scattering amplitude in the EFT for the massless pions, built as the $O(N+1)/O(N)$ σ-model, in [24]. Later the same method was employed for the calculation of LLog behavior of pion form factors [25].

In the present chapter we develop and demonstrate the method of calculation of LLogs at all orders in a massless EFT. As an example of an massless EFT we use the $O(N+1)/O(N)$ σ-model. First, we demonstrate the features of massless EFTs and the structure of its perturbative expansion. After that we obtain the recursive equation for LLog coefficients in the 4-point amplitude. We also derive the corresponding equations for the 6-point amplitude and for form factors. The presented method is also applied to the theories with other symmetries, in particular, to the massless ChPT and matrix model. Finally, we discuss methods of the solution of the obtained equations.

II.1 Property of a massless theory

In the renormalizable theories only some selected types of the diagrams diverge. The renormalization of these limited types of diagrams makes the theory finite. In contrast, in the non-renormalizable theories **all types** of the diagrams diverge. However, in the exceptional case of the massless theory some classes of the graphs

Figure II-1: The graph which gives the dressing of the propagator at 2-loops level, and its counterterm. The back blob denotes the counterterm vertex.

diverge weaker. This is a consequence of the fact that all diagrams with the massless tadpole subgraph are zero in the dimensional regularization[1].

To begin with, let us consider an example of the graph which does not contain the LLog term in the massless EFT. We take the graph which describes the dressing of the propagator in the ϕ^4-type EFT. The lowest contribution to the propagator dressing is of 2-loop order. The contribution is given by the only diagram shown in fig.II.1. From the naive consideration, the expression for the diagram should be of the form $\frac{p^6}{\varepsilon^2}\left(\frac{\mu^2}{p^2}\right)^{2\varepsilon} f(\varepsilon)$, and, therefore, $\ln^2(\mu^2)$ should appear in the finite part. But the direct computation shows that this diagram has only ε^{-1} pole and, hence, has no LLog contribution.

The absence of the leading pole in the graph in fig.II.1 can be described through the considering of the topology of graphs which perform the renormalization procedure. According to the rules of R$'$-operation, for the renormalization of the graph one has to compute the diagram with a drawn out loop. Then one subtracts it from the original graph. R$'$ operation eliminates the non-localities (logarithms) in the divergent part and allows one to perform the renormalization procedure by the appropriate redefinition of the couplings of the corresponding local operators. Schematically, the expression for the diagram in fig.II.1 after the R$'$-operation can by written in the following way

$$\frac{p^6}{\varepsilon^2}\left(\frac{\mu^2}{p^2}\right)^{2\varepsilon} f(\varepsilon) - \frac{A}{\varepsilon}\frac{p^6}{\varepsilon}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon} f_1(\varepsilon) = p^6\Big(\frac{f(0) - Af_1(0)}{\varepsilon^2} + \frac{2f'(0) - Af_1'(0)}{\varepsilon} + \text{Finite part}\Big),$$

where A is a renormalization constant of the 4-vertex. However, the single subtracted diagram for this graph has the tadpole topology (fig.II.1). Thus, the counterterm to the graph (fig.II.1) is equal to zero in the massless theory. Therefore, there is no subtraction term for the rigorous renormalization procedure. On the other hand the R-operation is well defined and counterterms to the local operators have to be local. The only possibility to satisfy this requirement for the diagram is to make it diverge as ε^{-1}, which confirms by the direct calculation.

Let us find the correlation between the graph topology and the graph power

[1]The final results of the section are independent of the regularization scheme, which is shown in the chapter III.

of divergence in the general form. We will use the recursive definition of the R′-operations (for the detailed description see e.g. [32],[30]) and the assumption that the resulting counterterm for the graph is local.

For the renormalization of a N_l-loop diagram one has to subtract from the diagram all possible subgraphs one by one. Note that in contrast to a renormalizable theory the diagrams in EFT do not contain subdivergences, i.e. every loop diverges at every order of subtraction. The subtraction of a loop in a graph generates a new (N_l-1)-loop graph. The pole part of the subtracted graph has to be added to the expression of the initial graph with the appropriate sign. This operation eliminates the non-locality from the subleading singular term ($\sim \varepsilon^{N_l-1}$) (the leading singular term ε^{N_l} is always local). In order to eliminate the non-localities from sub-subleading singularity one has to make the subtraction on the subtracted graph and add the result to the initial graph. Repeating this procedure until the tree level of the subtraction one obtains the local expression for the pole part of the diagram.

The set of all subgraphs to the graph forms an ordered graph space. For the complete R′-operation one has to pass through all possible paths in this space from the initial N_l-loop graph to the tree graph. Here it is convenient to introduce the notion of the subtraction length. The subtraction length is the number of possible graphs in a path without allowing steps on which tadpoles are subtracted, and also the initial graph is not taken into account. In these terms one can formulate the following rule: if for a N_l-loop diagram the maximum subtraction path length is l_s, then this diagram diverges as ε^{-l_s}. It can be proven in the following way. At the n_s-th step of the subtraction procedure one erases the non-locality from the $\varepsilon^{-(N_l-n_s)}$-th term of the initial graph. If some of the counter graphs do not exist (in any path of renormalization) the R′-operation leaves the non-local singular part, or the total divergence is less by unity. Thus, if $(N_l - l_s)$ counterterms do not exist, the total divergence is less or equal to l_s.

Let us consider a one-particle irreducible(1PI) graph $G_{N_l,n}$ with N_l loops and n vertices. Let W_k denotes the operation of the one loop subtraction, where the loop passes through $(k+1)$ vertices. W_k acting on a graph transforms it into the graph with the lower number of loops and vertices:

$$W_k G_{N_l,n} \mapsto G_{N_l-1,n-k}.$$

Note, that the subtraction of a tadpole subgraph is W_0. Passing through the full path of the subtractions one obtains the local vertex diagram $G_{0,1}$:

$$W_{k_1} W_{k_2} .. W_{k_l} G_{N_l,n} = G_{0,1}, \qquad \sum_{i=1}^{N_l} k_i = n-1. \tag{II.1-1}$$

If there is no tadpole subtraction in the path (II.1-1), i.e. $k_i \neq 0$ and $\sum_{i=1}^{N_l} k_i \geqslant N_l$, then the diagram $G_{N_l,n}$ contains the leading singularity and the LLog part. On the other hand the number of loops in the graph is given by

$$N_l = \sum_{j\geq 2}\Big(\frac{j}{2}-1\Big)n_j - \frac{N_\phi}{2} + 1, \tag{II.1-2}$$

where N_ϕ is the number of external lines and n_j is the number of the vertices with j incoming lines. Combining exp. (II.1-1) and eqn. (II.1-2) one obtains the following inequality

$$\frac{N_\phi}{2} - 2 \geqslant \sum_{j>2}\Big(\frac{j}{2}-2\Big)n_j. \tag{II.1-3}$$

This inequality gives the constraint on the classes of graphs that contain LLog contribution in the massless EFT. The particular examples of eqn. (II.1-3) application are given below:

- $N_\phi = 2$, the dressing of the propagator can not give the LLogs (exceptional case is the presence of the 3-vertices, which we do not consider now);
- $N_\phi = 4$, the 4-point amplitude can contain the LLogs only if $n_{j>4} = 0$, i.e. only 4-vertices are in the graph;
- $N_\phi = 6$, the 6-point amplitude can contain the LLogs only if $n_6 \leqslant 1$ and $n_{j>6} = 0$.

Thus, for the consideration of a 4-point Green function at the LLog accuracy, one needs only the 4-field part of the Lagrangian, which is infinitely smaller then the full Lagrangian.

The diagrams which contain the NLLogs contribution, satisfy the inequality

$$\frac{N_\phi}{2} - 1 \geqslant \sum_{j>2}\Big(\frac{j}{2}-2\Big)n_j. \tag{II.1-4}$$

In particular cases it gives:

- $N_\phi = 2$, the dressing of propagator can contain the NLLogs only if $n_{j>4} = 0$.
- $N_\phi = 4$, the 4-point amplitude can contain the NLLogs only if $n_6 \leqslant 1$ and $n_{j>6} = 0$.

and so on.

II.2 $O(N+1)/O(N)$ σ-model

As an example of an EFT we will often use the $O(N+1)/O(N)$ σ-model at $D = 4$. This is a simple non-renormalizable model with several advantages. The Lagrangian of the model has the form

$$\mathcal{L}_2 = \frac{1}{2}\big[\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi^a\partial^\mu\pi^a\big], \tag{II.2-5}$$

where the fields lie on the surface of S^{N+1}, i.e. $\sigma^2 + \sum_{a=1}^{N}\pi^a\pi^a = F^2$. Integrating out the σ-field one obtains the Lagrangian in the form (I.1-2)

$$\mathcal{L}_2 = -\frac{1}{2}\pi^a\partial^2\pi^a - \frac{1}{8F^2}\pi^2\partial^2\pi^2 - \frac{1}{16F^4}\pi^4\partial^2\pi^2 + \mathcal{O}(\pi^8). \tag{II.2-6}$$

This model contains all main features of an usual EFT, e.g. the dimensional coupling and the infinity row of the interaction terms with increasing number of fields even at the lowest order of the theory. Also this model has several advantages listed below.

- At $N = 3$, Lagrangian (II.2-5) is equivalent to the massless Weinberg Lagrangian [1], or to the massless two-flavor ChPT [34](compare with exp. (I.3-11)). The LLog structure of these theories is known up to the 2-loop order for the 4-point amplitude [42].
- At $N = 1$, the model (II.2-5) is a free field theory. Therefore, all corrections disappear. This feature can be used as a check in our future calculations.
- The model (II.2-5) has known solution at the large N limit, e.g. see refs. [44], [45]. This will provide the check for our calculations and allow us to access the accuracy of the 1/N expansion without tedious calculations.

The general structure of the higher order Lagrangians is the same as in ChPT, as it was discussed in section 1.2. Namely, due to the absence of masses, the only dimension bearers in the Lagrangian are derivatives. According to our convention of the notations we denote the lowest couplings from (II.2-6) as

$$g_{10} = \frac{1}{F^2}, \quad g_{10}^{(6)} = (g_{10})^2 = \frac{1}{F^4}, \quad \text{and so on.} \tag{II.2-7}$$

Here we have omitted the upper index for the 4-field operator couplings, and starting from here we do it everywhere if it makes no confusion. Also we are not going to write the auxiliary lower index until the place where it will have a sense.

Beta-functions Let us investigate RG-functions for the Lagrangian of the (II.2-6)-type. As it was discussed in the chapter I the bare coupling has the form

$$g_n^{(k)bare} = \mu^{\varepsilon(k-2)}\Big(g_n^{(k)}(\mu) - \delta Z_n^{(k)}(g(\mu))\Big), \tag{II.2-8}$$

where δZ is the coefficient near the operator in the corresponding counterterm. δZ contains the pole part of diagrams. For example, the first few terms $\delta Z_n^{(4)}$ have the

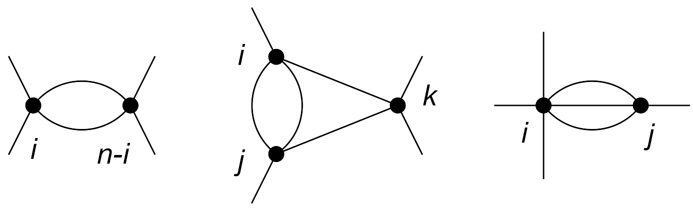

Figure II-2: The sample graph which contribute to the $\beta_n^{(4)}$, by the numbers the generation of vertices are denoted.

form

$$\delta Z_n^{(4)} = \underbrace{\frac{1}{\varepsilon}\sum_{i=1}^{n-1}\beta_{i,n-i}g_i g_{n-i}}_{\text{1-loop}} + \underbrace{\sum_{i+j+k=n}\left(\frac{b_2}{\varepsilon^2}+\frac{b_{1,a_1}}{\varepsilon}\right)g_i g_j g_k + \sum_{i+j=n-1}\frac{b_{1,a_2}}{\varepsilon}g_i^{(6)}g_j}_{\text{2-loop}} + \ldots \,. \tag{II.2-9}$$

The sample of diagrams, from which these counterterms come from, is shown in fig.II.2. Note that the last term in exp.(II.2-9), which corresponds to the last diagram in fig.II.2, has only single pole contribution due to exp. (II.1-3). The β-function in eqn. (I.4-22) is defined through the δZ as

$$\beta_n^{(k)} = \mu^2\frac{d\delta Z_n^{(k)}}{d\mu^2} + \varepsilon\delta Z_n^{(k)}. \tag{II.2-10}$$

The β-function has no singularities. Assuming the cancelation of the singularities in eqn.(II.2-10) one obtains that the β-function is composed of the simple pole coefficients:

$$\beta_n^{(k)} = -\sum_a b_{1a}\left(\frac{k_1+k_2+..+k_j-2(j+1)}{2}\right)\{g_{i_1}^{(k_1)}g_{i_2}^{(k_2)}...g_{i_j}^{(k_j)}\}_a \,, \tag{II.2-11}$$

where a enumerates all possible monoms of g's in $\delta Z_n^{(k)}$, b_{1a} are coefficients near the ϵ^{-1} in the structure a. The only constraint on the form of monoms is their momentum dimension. The monom dimension has to be the same as the momentum dimension of $g_n^{(k)}$. For example, from the consideration of eqn.(II.2-9) and eqn.(II.2-11) one obtains the first few terms for the β-function of the coupling $g_n^{(4)}$

$$\beta_n = \underbrace{-\sum_{i=1}^{n-1}\beta_{i,n-i}g_i g_{n-i}}_{1-loop} \underbrace{-2\sum_{i+j+k=n}b_{1,a_1}g_i g_j g_k - 2\sum_{i+j=n-1}b_{1,a_2}g_i^{(6)}g_j}_{\text{2-loop}} + \ldots \,. \tag{II.2-12}$$

In contrast to the renormalizable theories, an EFT contains an infinite number of β-functions. But every β-function is made of a finite number of terms, since $(n-1)$-loop diagrams can contribute only to β_k with $k \leqslant n$.

It is very convenient to introduce a general notation for all possible combinations of couplings g with similar properties. We will use the notation from the theory of sequences [52]. We denote all possible monoms of power j and total generation k using the following form:

$$|g^{\langle j \rangle}|_k = \sum_{i_1+i_2+..+i_j=k} g^{(a)}_{i_1} g^{(b)}_{i_2} ... g^{(c)}_{i_j}, \tag{II.2-13}$$

where the upper indices of g's are not fixed. Every term in the sum (II.2-13) enters with some uncontrolled number.

Using the notation (II.2-13), the expression for $\beta^{(4)}_n$ can be represented in the following form

$$\beta^{(4)}_n = -\sum_{l=1}^{n-1} \sum_{j=0}^{l-1} \sum_{p=0}^{n-l+j-1} |g^{\langle l+1-j+p \rangle}|_{n-j}, \tag{II.2-14}$$

where every term in the sum has an independent numerical factor. This factor can be found only from the direct loop calculation.

Let us explain the ingredients of exp. (II.2-14) and their origins in details. The index l enumerates the number of loops in the graph contribution. The upper limit for the sum over l is fixed by the following condition: to the counterterm of the $g^{(4)}_n$ only graphs with the number of loops less or equal to $(n-1)$ can contribute . The extreme case is reached when the graph has only $g^{(4)}$ vertices. Such case corresponds to the $|g^{\langle l+1 \rangle}|_n$–structure. Furthermore, one has to take into account the graphs with the vertices with a higher number of incoming lines. For a fixed number of loops such graphs have less vertices than those which contain only 4-vertices. The number of vertices is less by j, where j is a $\sum_{j=4}^{\infty} \frac{(j-4)}{2} n_j$, where n_j is the number of vertices with j incoming lines. On the other hand the total number of the interacting vertices can not be less than 2, otherwise it would be a pure tadpole diagram. This condition gives the restriction on the sum over j. The momentum dimension of the vertex with a higher number of incoming legs is higher (I.1-3). Thus, we decrease the lower index of the g-structure by j. Finally, one can add the $g^{(2)}$ vertices, which do not change the loop-number of graph, but contribute to the total dimension of β-function alike $g^{(4)}$. It happens because the $g^{(2)}_n$-vertex always comes into the graph together with one propagator, so effectively it has the same dimension as $g^{(4)}_n$. In the sum (II.2-14) the number of $g^{(2)}$-vertices is regulated by the summation index p. The upper limit of the sum over p is dictated by the constraint that the graph has to contain at least two non-$g^{(2)}$-vertices.

Analyzing in the same way the graphs with k-external lines one finds that the

β-function for the constant $g_n^{(k)}$ has the general form

$$\beta_n^{(k)} = -\sum_{l=1}^{n-1}\sum_{j=0}^{\frac{k}{2}+l-3}\sum_{p}|g^{\langle\frac{k}{2}-j+l-1+p\rangle}|_{n+\frac{k}{2}-j-2} \tag{II.2-15}$$

$$= -\sum_{l=1}^{n-1}\sum_{j=0}^{\frac{k}{2}+l-3}\sum_{p}|g^{\langle p+j+2\rangle}|_{n+j-l+1}.$$

The β-function has the same momentum dimension as the coupling to which the β-function relates. Thus, the dimensions of the coupling on the left-hand side of the exp. (II.2-15) have to be chosen in the appropriate way. This can be done using the upper indices of the couplings (II.2-13).

Considering the first generation of β-functions in the purely non-renormalizable theory one can find that there are no couplings which can be combined into the lowest dimension coupling. Consequently, from exp. (II.2-15) one has

$$\beta_1^{(k)} = 0. \tag{II.2-16}$$

Operator $\hat{H}$ and its properties The dimensionality of the couplings applies several restrictions on the constructions of type (II.2-13):

- The maximum generation of the coupling that takes part in the $|g^{\langle r\rangle}|_\alpha$ is $(\alpha-r+1)$. This situation occurs only if all others couplings are of the first generation,

$$|g^{\langle r\rangle}|_\alpha = \underbrace{g_1 g_1..g_1}_{r-1} g_{\alpha-r+1} + \dots \quad . \tag{II.2-17}$$

- If the structure $|g^{\langle r\rangle}|_\alpha$ has an momentum dimension d the maximum upper index of the coupling can be only $k = 4 - d - 2\alpha$;

- If the "power" and the total generation of the g-structure coincides, the g-structure contains only the couplings of the first generation,

$$|g^{\langle r\rangle}|_r = \underbrace{g_1^{(a)} g_1^{(b)}..g_1^{(c)}}_{r}. \tag{II.2-18}$$

- If the "power" is more than the total generation, the g-structure is zero identically,

$$|g^{\langle r\rangle}|_\alpha = 0, \quad r > \alpha. \tag{II.2-19}$$

The operator $\hat{H}$ is defined by exp.(I.4-24) together with exp. (II.2-15). Let us consider the action of $\hat{H}$ on a structure of type (II.2-13). The differentiation over a coupling and multiplication on g-structure transforms the g-structure to another g-structure in the following way

$$|g^{\langle r_1\rangle}|_{\alpha_1}\frac{d}{dg_n}|g^{\langle r_2\rangle}|_{\alpha_2}\sim|g^{\langle r_1+r_2-1\rangle}|_{\alpha_1+\alpha_2-n}, \qquad \text{(II.2-20)}$$

where under the sign $\sim$ the uncontrolled change of the numerical coefficients is understood. Therefore, acting by $\hat{H}$ on a g-structure one has

$$\hat{H}|g^{\langle r\rangle}|_\alpha\sim\sum_{n=2}^{\alpha-r+1}\sum_{k=2,4..}^{\infty}\sum_{l=1}^{n-1}\sum_{j=0}^{\frac{k}{2}+l-3}|g^{\langle j+p+r+1\rangle}|_{j-l+1+\alpha}. \qquad \text{(II.2-21)}$$

The upper limit of the summation over n is fixed by constraint (II.2-17).

The operator $\hat{H}$ is dimensionless, thus the energy dimension of the exp. (II.2-21) is the same as the dimension of $|g^{\langle r\rangle}|_\alpha$. But the number of couplings on the right-hand side of exp. (II.2-21) increases at least by one compared with the g-structure on the left-hand side. Hence, the maximum possible generation and the maximum possible upper index of the couplings is decreased according to constrain (II.2-17).

Acting a times by operator $\hat{H}$ on the g-structure one obtains

$$\hat{H}^a|g^{\langle r\rangle}|_\alpha\sim \qquad \text{(II.2-22)}$$
$$\sum_{n_1=2}^{\alpha-r+1}\sum_{l_1=1}^{n_1-1}\sum_{n_2=2}^{\alpha-r-l_1+1}\sum_{l_2=1}^{n_2-1}\cdots\sum_{n_a=2}^{\alpha-r-\sum_{i=1}^{a-1}l+1}\sum_{l_a=1}^{n_a-1}\sum_{k_i,j_i,p_i}|g^{\langle\sum_i(j_i+p_i)+r+a\rangle}|_{\sum_i(j_i-l_i)+\alpha+a},$$

where $\sum_i=\sum_{i=1}^a$. The sum over n_a can be non zero only in the case when

$$\alpha-r-1\geqslant\sum_{i=1}^{a-1}l_i\geqslant a-1, \qquad \text{(II.2-23)}$$

where for the second inequality the constrain $l_i\geqslant 1$ was used. Hence

$$\hat{H}^a|g^{\langle r\rangle}|_\alpha=0,\quad a>\alpha-r. \qquad \text{(II.2-24)}$$

Generally speaking eqn. (II.2-24) is the conjecture of exp. (II.2-16). Every application of the operator $\hat{H}$ increases the number of couplings in monoms keeping the total energy dimension unchanged. At some point, all the couplings in the expression are transformed into g_1, and the next action of the $\hat{H}$ turns the expression to zero according to exp. (II.2-16).

The maximum power of the $\hat{H}$, which does not give the zero result in exp. (II.2-22) is $a=\alpha-r$. In this case all l_i have to be unities. In other words, in this case one can leave only the one-loop part of the β-functions in the operator $\hat{H}$, since the appearance of any higher loop β-function gives the zero result. The g-structure on the right-hand side of exp. (II.2-22) for $a = \alpha - r$ is

$$|g^{\langle \sum_i (j_i+p_i)+\alpha \rangle}|_{\sum_i j_i+\alpha}.$$

According to exp. (II.2-19) all $p_i = 0$, and from exp. (II.2-18) follows that

$$\hat{H}^{\alpha-r}|g^{\langle r\rangle}|_\alpha = \hat{H}_1^{\alpha-r}|g^{\langle r\rangle}|_\alpha \sim g_1^{(a)} g_1^{(b)} ... g_1^{(c)}, \tag{II.2-25}$$

where the operator $\hat{H}_1$ is the $\hat{H}$ with one-loop contributions only . Note that $\hat{H}_1$ does not contain the $g^{(2)}$ vertices insertions. Somehow, we can say that the diagrams with p $g^{(2)}$ insertions have effectively $(l+p)$ number of loops.

There is a very intuitive graphical interpretation of the action of the operator $\hat{H}_1$ on the graph. The operator $\hat{H}_1$ slides apart the graph vertex and inserts on its place all possible one-loop subgraphs of the same dimension as the slided vertex. Thus, the tree graphs of order n can be transformed to the $(n-1)$-loop graphs by acting of the operator $\hat{H}_1^{n-1}$. In the next section solving the combinatorics of this transformation we will find a simple relation between LLog coefficients at different loop orders.

The structures like (II.2-25) are needed for the investigation of the LLog behavior of the diagram. For the consideration of NLLogs one has to consider eqn. (II.2-22) with $a = \alpha - r - 1$. The consideration of the constraints given by sums in exp. (II.2-22) and the constraint (II.2-19), gives three possibilities to obtain a non-zero answer, namely: when all $l_i = 1$ and $p_i = 0$; when one of $l_j = 2$ and $l_{i\neq j} = 1$, $p_i = 0$; when one of $p_j = 1$, and $l_i = 1$, $p_{i\neq j} = 0$. We denote by $\hat{H}_2$ the part of operator $\hat{H}$ with $l = 2$, $p = 0$ or with $l = 1$, $p = 1$. Analogously to exp. (II.2-25) we write

$$\hat{H}^{\alpha-r-1}|g^{\langle r\rangle}|_\alpha = \left(\hat{H}_1^{\alpha-r-1} + \sum_{j=1}^{\alpha-r-1} \hat{H}_1^j \hat{H}_2 \hat{H}_1^{\alpha-r-2-j}\right)|g^{\langle r\rangle}|_\alpha. \tag{II.2-26}$$

Thus, one has two completely different structures on the same logarithmical order

$$\sum_{j=1}^{\alpha-r-1} \hat{H}_1^j \hat{H}_2 \hat{H}_1^{\alpha-r-2-j}|g^{\langle r\rangle}|_\alpha \sim g_1^{(a)} g_1^{(b)} ... g_1^{(c)}, \tag{II.2-27}$$

$$\hat{H}_1^{\alpha-r-1}|g^{\langle r\rangle}|_\alpha \sim g_2^{(a)} g_1^{(b)} ... g_1^{(c)}. \tag{II.2-28}$$

In the next section we will discuss the connection between this structure and logarithmical structure of the matrix element in details . Here we only note that these two structures correspond to the different types of logarithm contributions. The logarithms, which come from the term (II.2-27), will be called "true" NLLog contributions, because they correspond to the real next-to-leading contributions of the $(n-1)$-loop graphs at the n-th order of expansion. One has to make a two-loop calculation in order to obtain "true" NLLogs. The logarithms, which come from the term (II.2-28), will be called "false" NLLog contributions, because they are the leading contribution of $(n-2)$-loop graphs at the n-th order of expansion.

The results of this section are general for any theory of ϕ^4-type. The obtained hierarchy, operator properties are based only on the counting of dimensions and on the relation (II.2-16).

II.3 LLog for $\pi\pi$ scattering

Let us consider the 4-point amplitude in the $O(N+1)/O(N)$ σ-model. We call it $\pi\pi$ scattering because at $N=3$ the model (II.2-6) coincides with the two-flavor ChPT, which describes the pion interactions.

The g-structure of the n-th order of the chiral expansion for the $\pi\pi$ scattering amplitude is similar to the β-function of the $g_n^{(4)}$ (II.2-14). One has to add only the tree diagram term, i.e. with $l=0$ and $j=l$. The coefficient instead of every g-structure contains all powers of $\ln\mu^2$ less or equal to the number of loops, which is enumerated by l index of summation. Combining together expressions (I.1-8) and (II.2-14), and the explanations of section II.1 one obtains the general structure of the 1PI graphs that describe the $\pi\pi$ scattering matrix element:

$$\begin{aligned}\mathcal{M}_{\pi\pi} &= \langle\pi^a\pi^b|T|\pi^c\pi^d\rangle \\ &= \sum_{n=1}^{\infty} E^{2n}\sum_{l=0}^{n-1}\sum_{j=0}^{l}\sum_{p=0}^{n-l+j-1}|g^{\langle l+p+1-j\rangle}|_{n-j}\sum_{m=0}^{l-j}\ln^m(\mu^2)f_{n,j,l,p}(s,t),\end{aligned} \tag{II.3-29}$$

where s and t are Mandelstam variables, f is a dimensionless function of its arguments and $T=i(S-I)$. We remind that this expansion is valid only in the region $s\sim t\ll F^2$.

The LLog terms in the amplitude $\mathcal{M}_{\pi\pi}$ are the terms proportional to $E^{2n}\ln^{n-1}(\mu^2)$. According to exp. (II.2-18) and the power counting the LLog term is proportional to $(g_1^{(4)})^n$. The zero-loop contribution of the amplitude (II.3-29) consists of $|g^{\langle 1\rangle}|_n$, which is simply $g_n^{(4)}$. In principle the tree diagrams with $g^{(2)}$-vertices on the external legs have to be presented in exp. (II.3-29). But these diagrams do not give a LLog contribution, and we do not consider them.

In order to extract the LLog coefficient, one differentiates the coefficient near the E^{2n} in the amplitude (II.3-29) other $\ln(\mu^2)$ $(n-1)$-times,

$$\begin{aligned}&\left(\mu^2\frac{d}{d\mu^2}\right)^{n-1}\sum_{l=0}^{n-1}\sum_{j=0}^{l}\sum_{p=0}^{n-l+j-1}|g^{\langle l+p+1-j\rangle}|_{n-j}\sum_{m=0}^{l-j}\ln^m(\mu^2)f_{n,j,l,p}(s,t) \\ &\qquad = (n-1)!f_{n,n-1}(s,t)|g^{\langle n\rangle}|_n = (n-1)!f_{n,n-1}(s,t)\left(g_1^{(4)}\right)^n.\end{aligned} \tag{II.3-30}$$

According to eqn. (I.4-23) the differentiating over $\ln\mu^2$ is equivalent to the action of the operator $\hat{H}$ on the same coefficient. Using constraint (II.2-24) and relation (II.2-25) one obtains

$$\hat{H}^{n-1}\sum_{l=0}^{n-1}\sum_{j=0}^{l}\sum_{p=0}^{n-l+j-1}|g^{\langle l+p+1-j\rangle}|_{n-j}\sum_{m=0}^{l-j}\ln^m(\mu^2)f_{n,j,l,p}(s,t) \\ = f_{n,0}(s,t)\hat{H}_1^{n-1}|g^{\langle 1\rangle}|_n = f_{n,0}(s,t)\hat{H}_1^{n-1}g_n^{(4)}. \tag{II.3-31}$$

The functions $f_{n,0}(s,t)$ in exp. (II.3-31) come directly from the pure 4-field part of the $\mathcal{L}_n$, i.e. it is the momentum representation of the Feynman rule for the 4-field operators in $\mathcal{L}_n$. The $f_{n,0}$ can be expanded over some complete basis P_C in the momentum and group spaces. The coefficient of every term can be chosen as unities with successful redefinition of the corresponding coupling, i.e.

$$f_{n,0}(s,t)g_n^{(4)} = \sum_C P_C(s,t)g_{nC}^{(4)}. \tag{II.3-32}$$

On the Lagrangian level this operation satisfies the expansion of the $\mathcal{L}_n$ onto the set of the independent operators, see exp. (I.4-18). The expansion over the same basis for exp. (II.3-30) has the form

$$f_{n,n-1}\big(g_{10}^{(4)}\big)^n = \sum_C \omega_{nC}\big(g_{10}^{(4)}\big)^n P_C(s,t). \tag{II.3-33}$$

The ω_{nC} is the desired LLog coefficient. Since the basis P_C is complete, one can compare structures near every P_C independently. From the equality of expressions (II.3-30) and (II.3-31), follows that the LLog coefficients ω_{nC} are

$$\omega_{nC} = \frac{\big(g_{10}^{(4)}\big)^{-n}}{(n-1)!}\hat{H}_1^{n-1}g_{nC}^{(4)}. \tag{II.3-34}$$

The most right operator $\hat{H}_1$ in definition (II.3-34) acts only on the $g_n^{(4)}$, thus it can be presented in the form (omitting the auxiliary index)

$$\hat{H}_1 = \sum_{i=1}^{n-1}\beta_{i,n-i}g_i^{(4)}g_{n-i}^{(4)}\frac{\partial}{\partial g_n^{(4)}}, \tag{II.3-35}$$

where $\beta_{i,n-i}$ is the coefficient near the pole in the left diagram in fig.II.2. And again it contains only $g^{(4)}$ couplings, hence all $\hat{H}_1$ in definition (II.3-34) has the form of (II.3-35).

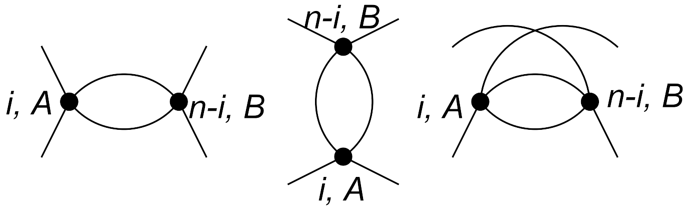

Figure II-3: The diagrams needed for the calculation of $\beta(i,A;n-i,B/C)$.

Using the exp. (II.3-35) one can write the chain of equalities

$$\begin{aligned}\omega_n &= \frac{g_1^{-n}}{(n-1)!}\hat{H}_1^{n-2}\sum_{i=1}^{n-1}\beta_{i,n-i}g_ig_{n-i}\\ &= \frac{g_1^{-n}}{(n-1)!}\sum_{i=1}^{n-1}\beta_{i,n-i}\sum_{k=0}^{n-2}\binom{n-2}{k}\left[\hat{H}_1^k g_i\right]\left[\hat{H}_1^{n-k-2}g_{n-i}\right].\end{aligned}$$

Eqn. (II.2-24) constrains the summation other k to value $k=i-1$ only. Using the definition of ω_{nC} in the right-hand side one obtains the recursive equation for the LLog coefficient (we restore the auxiliary indices):

$$\omega_{nC} = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{A,B}\beta(i,A;n-i,B/C)\omega_{iA}\omega_{n-i,B}, \tag{II.3-36}$$

where the $\beta(i,A;,n-i,B/C)$ is the coefficient near P_C/ε of the diagram in fig.II.3.

Eqn. (II.3-36) allows one to restore all coefficients ω_{nC} starting from the initial point. The natural initial value is ω_{1C} – the coefficient near the P_C structure at the first order of the expansion. The constants g_{1C} can be always redefined in such a way that $\omega_{10}=1$, and all other constants are zero.

Eqn. (II.3-36) is a conjecture of the topological properties of one-loop graphs of the theory. Therefore, its form is a general form of equation for the LLog coefficient for the 4-point amplitude in the theory with leading ϕ^4 interaction. The recursive equation for the LLogs in renormalizable theories, see e.g. [29], are particular cases of eqn. (II.3-36).

It is a non-linear recursive equation, and later we will show that it is equivalent to a non-linear integral equation in form of Hammerstein. The regular method of its solving does not exist. The perturbative approach, due to the non-linearity of the equation, has an uncontrollable error and usually gives a very bad approximation. On the other hand, eqn. (II.3-36) is easy and fast numerically solved , as it will be discussed at the end of this chapter.

Higher order Lagrangian The next step is the calculation of the $\beta(i,A;n-i,B/C)$. It is given by the coefficient near the pole of diagrams shown in fig. II.3.

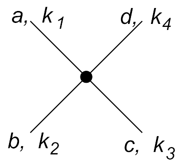

Figure II-4: The notation of Feynman rule for the 4-π vertex. All momenta are incoming.

The main problem is that the diagrams contain the 4-vertices from the arbitrary high order Lagrangian, which is unknown. The idea of solving this problem is following: since the values of higher order constants are out of interest, one can build the higher order Lagrangian from all possible independent operators.

The initial Lagrangian (II.2-6) has only one operator with 4 pion fields

$$V_1 = \frac{g_{10}}{8}\pi^2\partial\pi^2. \tag{II.3-37}$$

The Feynman rule for vertex V_1 has the form (the order of indices and momenta is shown in fig.II.4)

$$V_{10}^{abcd}(k_1,k_2,k_3,k_4) = g_{10}\Big[\delta^{ab}\delta^{cd}(k_1+k_2)^2 + \delta^{ac}\delta^{bd}(k_1+k_3)^2 + \delta^{ad}\delta^{bc}(p_1+p_4)^2\Big].$$

Direct calculation of the $V_1 * V_1$ loop diagram shows that the pole coefficient contains two types of structures: $\delta^{ab}\delta^{cd}(k_1+k_2)^2$ and $\delta^{ab}\delta^{cd}(k_1+k_4)^2$. All other structures relate to these two by crossing transformations. Thus, the next order Lagrangian has two independent vertex operators:

$$V_2 = \frac{g_{20}}{8}\pi^2[\partial^2]^2\pi^2 + \frac{g_{22}}{8}[\pi^a\nabla_\mu\nabla_\nu\pi^a][\pi^b\nabla_\mu\nabla_\nu\pi^b], \quad \nabla_\mu = \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}. \tag{II.3-38}$$

For $N=3$ case the couplings g_{20} and g_{22} are given by a known linear combination of l_1 and l_2 from Lagrangian (I.3-13), see (I.3-16).

The next generation of operators follows from the consideration of the $V_2 * V_1$ diagram and so on. At the n-th order the 4-field operators in the Lagrangian can be presented in the form

$$V_n = \sum_{C=0,2,.}^{n} V_{nC} = \sum_{C=0,2..}^{n} \frac{g_{nC}}{8}[\pi^a\nabla_{\mu_1}..\nabla_{\mu_C}\pi^a](\partial^2)^{n-C}[\pi^b\nabla_{\mu_1}..\nabla_{\mu_C}\pi^b]. \tag{II.3-39}$$

The expression (II.3-39) does not contain terms like $\partial^2\pi^a$ because their Feynman rule is proportional to p^2 and gives zero contribution to the amplitude at the one-loop level. In other words, these operators are proportional to the higher-field operators due to the equation of motions for the massless Lagrangians of type-(II.2-6): $\partial^2\pi^a \sim$

$\pi^a\pi^2$. The index C obtains only even values since π-fields are boson fields.

Generally speaking, exp. (II.3-39) corresponds to all possible not forbidden independent compositions of the derivatives between four fields. This basis is very inconvenient for the calculation. We found that it is useful to use as the basis the following linear combination of operators (II.3-39):

$$V_{nC}^{new} = \frac{g_{nC}}{8} \sum_{k=0,2,..}^{C} (-1)^{\frac{C-k}{2}} \frac{\Gamma\left(\frac{C+k}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{C-k}{2}+1\right)} \frac{V_{nk}}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+1\right)}. \qquad \text{(II.3-40)}$$

The sum on the right-hand side is nothing else as the Legendre polynomial expansion. Therefore we denote

$$V_{nC}^{new} = \frac{g_{nC}}{8}\pi^2(\partial^2)^n P_C\left(\frac{\nabla_1\nabla_2}{\partial^2}\right)\pi^2, \qquad \text{(II.3-41)}$$

where ∇_1 acts only between the left pair of fields and ∇_2 only between the right pair of fields. Starting from here we will use only this operators as a basis, therefore, we omit the label "new".

The Feynman rule for this operator is

$$V_{nC}^{abcd}(k_1,k_2,k_3,k_4) \;=\; g_{nC}\left[\delta^{ab}\delta^{cd}A_{nC} + \delta^{ad}\delta^{bc}B_{nC} + \delta^{ac}\delta^{bd}C_{nC}\right], \qquad \text{(II.3-42)}$$

$$A_{nC} \;=\; i(-1)^{n+1}[(k_1+k_2)^2]^n P_C\left(\frac{(k_1-k_2)\cdot(k_3-k_4)}{(k_1+k_2)^2}\right),$$

$$B_{nC} \;=\; i(-1)^{n+1}[(k_1+k_4)^2]^n P_C\left(\frac{(k_1-k_4)\cdot(k_2-k_3)}{(k_1+k_4)^2}\right),$$

$$C_{nC} \;=\; i(-1)^{n+1}[(k_1+k_3)^2]^n P_C\left(\frac{(k_1-k_3)\cdot(k_2-k_4)}{(k_1+k_3)^2}\right).$$

The basis of operators (II.3-41) is complete in the momentum space. We will use it without any changes in all other EFTs of ϕ^4-type with needed expansion of the symmetry structures.

The amplitude (II.3-29) can be expanded on three structures of group indices (we use the convention for $\pi\pi$-scattering notation as in [46], [47])

$$\mathcal{M} = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ad}\delta^{bc}B(s,t,u) + \delta^{ac}\delta^{bd}C(s,t,u), \qquad \text{(II.3-43)}$$

where the amplitudes A, B and C are related to each other by the crossing symmetry

$$B(s,t,u) = A(t,u,s), \quad C(s,t,u) = A(u,s,t).$$

The LLog part of the momenta expansion of the $A(s,t,u)$ in the basis (II.3-42) has the form

$$A(s,t,u) = (4\pi)^2 \sum_{n=1}^{\infty}\sum_{C=0}^{n} \hat{S}^n P_C\left(1+\frac{2t}{s}\right)\omega_{nC}\ln^{n-1}\left(\frac{\mu^2}{|s|}\right) + \mathcal{O}(\text{NLLog}), \quad \text{(II.3-44)}$$

where $\hat{S} = \frac{s}{(4\pi F)^2}$. Here, we put $|s|$ in the argument of the logarithm just from dimension reasons. The tree calculation with the lowest order Lagrangian gives $\omega_{10} = 1$.

We will often call A $(B,\ C)$-amplitudes just as $s(t,\ u)$-channel amplitudes, since their arguments of the Legendre polynomials are cosines of the scattering angle in $s(t,\ u)$-channels. Thus, the coefficient ω_{nC} has a meaning of the LLog coefficient of the n-chiral order expansion of $t_C(s)$ partial wave.

The calculation of $\beta(i,A;n-i,B/C)$ The loop-contraction of two identical vertices V_{nC} in the corresponding channel is diagonal, i.e.

$$A_{iA} * A_{n-i,B} = \int d^D l \frac{A_{iA}(p_1,p_2,l-P,-l)A_{n-i,B}(l,P-l,p_3,p_4)}{l^2(P-l)^2} \sim \frac{1}{\varepsilon}\frac{\delta^{AB}}{2A+1}A_{nA}. \quad \text{(II.3-45)}$$

The details of the calculation can be found in Appendix A.1. In some sense the operators (II.3-41) are the one-loop representation of the 4-field conformal basis.

The one-loop convolution of A and B (or C)-structures is not so simple. Such convolutions can be taken down to (II.3-45), using the completeness of the Legendre polynomial basis. If the external legs of the graph (fig.II.3) are on the mass-shell one can perform the expansion

$$B_{nA} = \sum_{B=0}^{n}(-1)^A\Omega_n^{AB}A_{nB}, \quad \text{(II.3-46)}$$

$$C_{nA} = \sum_{B=0}^{n}(-1)^B\Omega_n^{AB}A_{nB}, \quad \text{(II.3-47)}$$

where Ω is $(A,B\leqslant n)$

$$\Omega_n^{AB} = \frac{(2B+1)}{2^{n+1}}\int_{-1}^{1}dx\ P_A\left(\frac{x+3}{x-1}\right)(x-1)^n P_B(x). \quad \text{(II.3-48)}$$

Also we assume that $\Omega_n^{AB}=0$, if $A,B>n$.

One can say that Ω_n is a $(n+1\times n+1)$ matrix with a matrix element given by exp. (II.3-48). If it makes no confusion we will use the matrix notation. The factors $(-1)^A$ will be presented by the matrix U which is a diag$\{1,-1,1,..\}$ matrix of the corresponding dimension. In such notations expressions (II.3-46) and (II.3-47) are

$$B_n = U\Omega_n A_n, \quad C_n = \Omega_n U A_n.$$

The matrix Ω is the crossing matrix in the space of the partial waves, i.e. it transforms the coefficient near s^n in expansion of the s-channel partial wave $t_l(s)$ to the coefficient of the $t-$(or $u-$)channel partial wave. The crossing matrices Ω and U satisfy the following properties

$$\Omega_n\Omega_n = I, \quad \Omega_n U \Omega_n = U\Omega_n U, \tag{II.3-49}$$

where I is the unit matrix. Therefore, the set $\{I,\ U\Omega,\ \Omega U\}$ forms a group of the triangle rotation under the matrix multiplication, called C_3-group. Other properties of the Ω_n and its explicit expression can be found in Appendix B.1.

Using the matrix Ω, one can expend the subamplitudes B and C in amplitude (II.3-43) through s-channel partial waves:

$$B(s,t,u) = A(t,u,s) = (4\pi)^2 \sum_{n=1}^{\infty} \hat{S}^n \omega_n U\Omega_n P\left(1+\frac{2t}{s}\right) \ln^{n-1}\left(\frac{\mu^2}{|s|}\right) + \mathcal{O}(\text{NLLog}), \tag{II.3-50}$$

$$C(s,t,u) = A(u,s,t) = (4\pi)^2 \sum_{n=1}^{\infty} \hat{S}^n \omega_n \Omega_n U P\left(1+\frac{2t}{s}\right) \ln^{n-1}\left(\frac{\mu^2}{|s|}\right) + \mathcal{O}(\text{NLLog}). \tag{II.3-51}$$

Using these relations and basic integral (II.3-45) one obtains the β-function:

$$\beta(i,A;n-i,B/C) = \tag{II.3-52}$$
$$\left[\frac{N}{2}\frac{\delta_{AB}\delta_{AC}}{2C+1} + \frac{\delta_{AC}\Omega^{BA}_{n-i} + \delta_{BC}\Omega^{AB}_{i}}{2C+1} + 2\frac{1+(-1)^C}{2}\sum_{J=0}^{\min[i,n-i]} \frac{\Omega^{AJ}_i\Omega^{BJ}_{n-i}\Omega^{JC}_n}{2J+1}\right].$$

The details of the calculation can be found in Appendix A.2.

The investigation of the equation Substituting β-function (II.3-52) into eqn. (II.3-36) one can generate the coefficients ω_{nC} up to an arbitrary order. The analytical solution of eqn. (II.3-36) has not been found yet. The numerical solution was obtained using the package *Mathematica 7.0.* The calculation of the coefficients up to 20-th order with arbitrary N takes dozen seconds on an average computer. At the fixed N, the calculation is much faster. For example, the calculations of coefficients up to 20-th order take ~ 2 seconds, and up to 80-th order take about four minutes. In table II.1 the first few values of the ω_{nC} are listed. The one- and two-loop values coincide with the values obtained by the explicit calculation performed in [46],[47].

Table II-1: First few values of LLog coefficients ω_{nC} in $\pi\pi$ scattering in the $O(N+1)/O(N)$ σ-model.

$n \setminus C$	0	2	4
1	1		
2	$\frac{N}{2}-\frac{17}{18}$	$\frac{1}{9}$	
3	$\frac{N^2}{4}-\frac{79N}{144}+\frac{31}{48}$	$\frac{5N}{144}+\frac{5}{144}$	
4	$\frac{N^3}{8}-\frac{2207N^2}{5400}+\frac{107137N}{194400}-\frac{80719}{194400}$	$\frac{131N^2}{15120}+\frac{131N}{7776}-\frac{1189}{272160}$	$\frac{N^2}{700}-\frac{29N}{10800}+\frac{361}{75600}$
5	$\frac{N^4}{16}-\frac{1313N^3}{5400}+\frac{3547363N^2}{7776000}-\frac{1723427N}{3888000}+\frac{46531}{172800}$	$\frac{83N^3}{40320}+\frac{823N^2}{67200}-\frac{200903N}{10886400}+\frac{91849}{2177280}$	$\frac{N^3}{1120}-\frac{9089N^2}{6048000}+\frac{143893N}{54432000}+\frac{1451}{5443200}$

Table II-2: First few numerical values of LLog coefficients ω_{nC} in $\pi\pi$ scattering in two-flavor ChPT

$n \setminus C$	0	2	4	6	8	10	12
1	1						
2	0.55	0.111					
3	1.25	0.139					
4	0.93	0.124	9.58×10^{-3}				
5	1.54	0.153	1.88×10^{-2}				
6	1.34	0.118	2.73×10^{-2}	7.04×10^{-4}			
7	1.95	0.140	3.73×10^{-2}	1.93×10^{-3}			
8	1.85	0.110	3.93×10^{-2}	3.83×10^{-3}	5.00×10^{-5}		
9	2.49	0.125	4.97×10^{-2}	6.12×10^{-3}	1.78×10^{-4}		
10	2.51	0.102	4.80×10^{-2}	8.23×10^{-3}	4.41×10^{-4}	3.54×10^{-6}	
11	3.23	0.114	5.79×10^{-2}	1.14×10^{-2}	8.22×10^{-4}	1.56×10^{-5}	
12	3.36	0.095	5.46×10^{-2}	1.33×10^{-2}	1.33×10^{-3}	4.59×10^{-5}	2.53×10^{-7}
13	4.20	0.105	6.38×10^{-2}	1.72×10^{-2}	2.06×10^{-3}	9.86×10^{-5}	1.32×10^{-6}
14	4.48	0.090	6.00×10^{-2}	1.88×10^{-2}	2.80×10^{-3}	1.86×10^{-4}	4.50×10^{-6}
15	5.49	0.099	6.87×10^{-2}	2.32×10^{-2}	3.93×10^{-3}	3.20×10^{-4}	1.10×10^{-5}

The presented form is not intuitive. In order to show the numerical tendency of the ω's values we present the first few ω_{nC} at $N=3$, (table II.2) . From table II.2 one can see that $\omega_{nC} \sim a^n 10^{-C/2}$, where a is some number. Therefore, the series (II.3-44) has a non-zero radius of convergence. The asymptotic behavior $\omega_n \sim a^n$ also follows from the homogeneity of eqn. (II.3-36) (see details in appendix C.2). The coefficient a can be found numerically. In case of $N=3$ considering of the first 80-loops we found that

$$\omega_{n0}\Big|_{n\to\infty} \sim 0.76(1.15)^{n-1}, \quad \sum_C \omega_{nC}\Big|_{n\to\infty} \sim 0.78(1.15)^{n-1}.$$

The contribution of the higher partial waves to the LLog approximation of scattering amplitude is numerically very small.

At $N=1$, the theory (II.2-5) is a free theory, and there are no quantum corrections. We have not paid any attention to this possibility yet. The β-function has been calculated under the assumptions that there exists an interaction of all

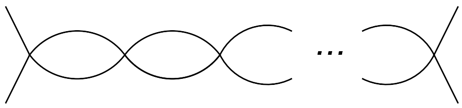

Figure II-5: The chain diagrams gives the leading large-N order to $\pi\pi$-scattering.

orders. At $N = 1$, the Kronecker deltas in the expression (II.3-43) are unites, and the amplitude is the sum of amplitudes in different channels:

$$\mathcal{M} = A(s,t,u) + B(s,t,u) + C(s,t,u) \sim \sum_{n=1}^{\infty} \hat{S}^n \omega_n (I + U\Omega_n + \Omega_n U).$$

Although all ω_{nC} generated by eqn (II.3-36) at $N = 1$ are non-zero, the particular combination $\omega_n(I + U\Omega_n + \Omega_n U)$, which only appears in the amplitude, is zero at all orders.

The large-N limit of the theory (II.2-5) is well known, e.g. [43],[44]. The leading order of the large-N expansion is given by the bubble-chain diagram (fig.II.5). The higher orders are given by the diagrams with several bubble-chain subdiagrams. The number of chains is the level of the large-N expansion. Every loop of the chain brings the factor $\frac{N}{2}$ at large-N limit and a chain of $(n-1)$ loops is equal to $s^n \left(\frac{N}{2}\right)^{n-1} I^{n-1}(\varepsilon)$, where $I(\varepsilon)$ is the scalar loop integral. After the R-operation the LLog coefficient becomes $\left(\frac{N}{2}\right)^{n-1}$.

The calculation of the next-to-leading order of large-N expansion is a difficult task, even in LLog approximation. One has to solve the combinatoric of the R-operation for a $(n_1 + n_2 + 1)$-loop diagram. Here the number of loops is calculated as the sum of loops in the first (second) chain and one loop which connecting the chains. The usage of eqn. (II.3-36) allows one to find the corrections to the large-N expansion without difficult calculations.

The ω_{nC} contains all powers of N up to N^{n-1} order, which is also seen in table II.1. Thus, the solution of eqn. (II.3-36) can be represented as follows

$$\omega_{nC} = N^{n-1}\omega_{nC}^{(0)} + N^{n-2}\omega_{nC}^{(1)} + .. + \omega_{nC}^{(n-1)}. \tag{II.3-53}$$

the β-function (II.3-52) is separated into two parts

$$\beta(i, A; n-i, B/C) = N\beta_0 + \beta_1, \quad \beta_0 = \frac{N\delta^{AC}\delta^{BC}}{2(2C+1)}.$$

The equation for the leading N term, $\omega^{(0)}$, has a simple form

$$\omega_{nC}^{(0)} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \frac{\omega_{iC}^{(0)}\omega_{n-i,C}^{(0)}}{2C+1}, \tag{II.3-54}$$

and the simple solution

$$\omega_{nC}^{(0)} = \delta^{C0} \left(\frac{1}{2}\right)^{n-1}, \tag{II.3-55}$$

which fully agrees with the regular calculation [44]. Note, that all $\omega_{n,C\neq 0}$ are next-to-leading in large-N counting, which is also seen in table 2.1. Thus the s-channel amplitude at leading large-N order is independent of Mandelstam variables t and u,

$$A_{\text{Large-N}}(s,t,u) = \frac{s}{F^2\left(1 - \frac{N}{2}\hat{S}\ln\left(\frac{\mu^2}{s}\right)\right)}.$$

One can find higher order $\omega^{(k)}$ recursively, (the details of the solution can be found in Appendix C.1),

$$\omega_{nC}^{(k)} = f_C^{(k)}(n) + \delta_{C0}\sum_{m=1}^{n-1}\frac{2n}{(m+1)m}f_0^{(k)}(m), \tag{II.3-56}$$

where

$$f_C^{(k)}(n) = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{k-1}\frac{\omega_{iC}^{(j)}\omega_{n-i,C}^{(k-j)}}{2C+1} + \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{A,B=0}^{i,n-i}\beta_1\sum_{j=0}^{k-1}\omega_{iA}^{(j)}\omega_{n-i,B}^{(k-j-1)}.$$

II.4 Form factors in LLog approximation

Scalar form factor Form factors (FFs) are very important objects in particle physic. They give information about the structure of particles and effectively enter into many phenomenological calculations. Also FFs seem to be the simplest object for the consideration in EFTs. They are often used for the normalization of low energy constants (LECs) in EFTs.

The simplest form factor in the model (II.2-5) is the scalar form factor of the pion. It is defined as the matrix element of the scalar isospin-0 current between the vacuum and the two-pion states, i.e.

$$\langle 0|J(0)|\pi^a(p_1)\pi^b(p_2)\rangle = \delta^{ab}F(s), \tag{II.4-57}$$

where $s = (p_1+p_2)^2$. The current $J(0)$ can be represented as a whole set of scalar isospin-0 operators, which can be constructed from the building blocks of our theory, i.e. pion fields π^a and derivatives. The only constraint on the construction of the current operator is the symmetries of the theory and the current. Every independent operator in the current comes with its own normalization constant. The lowest order operator in this expansion contains no derivatives. It can be written as

$$J(0) = \tilde{B}(\sigma^2 + \frac{\pi^2}{F^2}) \simeq B\pi^2 + \mathcal{O}(\pi^4), \tag{II.4-58}$$

where B is some constant.

The tree calculation of the matrix element (II.4-57) gives us $F(s) = 2B + \mathcal{O}(s)$. At $N = 3$, when the model coincides with 2-flavor ChPT, the normalization constant B is

$$B \simeq \frac{m_\pi^2}{m_q},$$

where m_π is the lowest pion mass, and $m_q = m_u = m_d$ is the mass of light quarks. The quantum correction to the scalar FF in ChPT is known up to two-loop order [48] from the direct calculation. The LLog coefficients are known up to four-loop order [22]. So, we can carry out a strong check for the method.

The dimension analysis says that the perturbative expansion can be written in the form

$$F(s) = 2B(4\pi)^2 \sum_{n=0}^{\infty} \hat{S}^n \sum_{k=0}^{n} \ln^k \left(\frac{\mu^2}{-s}\right) T_{n,k}, \tag{II.4-59}$$

where $\hat{S} = \frac{s}{(4\pi F)^2}$, and $T_{0,0} = 1$. Note that the matrix element (II.4-57) is a one-parametric object, i.e. only one dimensional parameter s is presented. The analytic properties of the form factor in the complex s-plane are very simple. In the massless limit, form factors have only cuts along real $s > 0$ ray. Thus, the only argument that can appear in logarithms is $(-s)$. Our aim is to find the LLog behavior of the $F(s)$, i.e. coefficients $T_{n,n} = v_n$.

The method of obtaining the coefficients v_n is the same as we had for the $\pi\pi$ scattering. One has to examine the renormalization properties of the external composite operator (II.4-58) in the theory, and properly define the LLog coefficient through the operator $\hat{H}$. The consideration of the definition gives the recursive equation for the LLog coefficient v_n. Here, we list the main features of the FF considerations without their proofs, since the proofs are very similar to the previous ones.

- Multi-pion operators (with more then 2-fields) do not contribute to FFs LLogs. And this statement remains true in any massless theory.

- In order to obtain LLog coefficient, one needs to consider only the one-loop renormalization of all order terms of the corresponded currents. For that the 2-field current operator have to be introduced at all order with unknown constants. For example, in the case of the scalar form factor the simplest choice

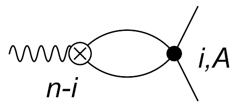

Figure II-6: The diagram which gives the one-loop anomalous dimension for the current operator.

is

$$J(0) \;=\; \sum_{n=0}^{\infty} f_n \frac{1}{2}\partial^{2n}\pi^2 + \mathcal{O}(\pi^4) \qquad \text{(II.4-60)}$$

with $f_0 = 2B$. Whereupon, one finds anomalous dimensions of every f_n, i.e. $\mu^2 \frac{df_n}{d\mu^2}$. We call the object $\mu^2 \frac{df_n}{d\mu^2}$ as anomalous dimension in analogy to the renormalizable theories, but to be precise it is the β-function of constants f. In the theory with leading ϕ^4 interaction the one-loop anomalous dimension has the form

$$\mu^2 \frac{\partial}{\partial \mu^2} f_n \;=\; -\sum_{i=0}^{n-1}\sum_{A=0}^{i} Z_1^{n,i,A} f_{n-i} g_{iA} + \mathcal{O}(f g^{\langle 2\rangle}), \qquad \text{(II.4-61)}$$

where Z_1 is a coefficient near the simple pole of the diagram in fig.II.6 projected on the f_n-operator. Note that the anomalous dimension with the corresponding differentiation also has to be added to the operator $\hat{H}$, i.e. $\hat{H} \to \hat{H} + Z_n[f,g]\frac{\partial}{\partial f_n}$. The initial order operator has no anomalous dimension, $Z_1 = 0$.

- Acting by the operator $\hat{H}^n$ on the perturbative series and making the similar analysis, as in eqn. (II.3-30,II.3-31) one finds that the LLog coefficient of the FF can be expressed as

$$v_n = \frac{(g_{10})^{-n}}{f_0} \frac{\hat{H}_1^n}{n!} f_n. \qquad \text{(II.4-62)}$$

Thus, v_n can be found recursively through the equation

$$v_{nC} \;=\; \frac{1}{n} \sum_{i=0}^{n-1}\sum_{A=0}^{n-i} Z_1^{n,i,A} v_i \omega_{n-i,A}. \qquad \text{(II.4-63)}$$

The boundary condition for eqn. (II.4-63) can be chosen as $v_n = 1$, by the proper redefinition of f_0. The ω_{nC} is the LLog coefficient of the $\pi\pi$ scattering amplitude (II.3-34) which satisfies eqn. (II.3-36).

The calculation of the $Z_1^{n,i,A}$ can be easily done with the technique that was introduced above. The operators (II.4-60) are the "half" (without left pair of π-fields) of the operator V_{n0} (II.3-41) and its anomalous dimension is closely connected with $\beta(i, A; n-i, 0/0)$. The details of the calculation are presented in Appendix A.3. The result is

$$Z_S^{n,i,A} = \frac{N}{2}\delta_{A0} + \frac{1+(-1)^A}{2}\Omega_{n-i}^{A0}, \tag{II.4-64}$$

where Ω_n^{AB} is defined by exp. (II.3-48), and particularly

$$\Omega_n^{A0} = \frac{(-1)^{n+A}(n-A)!(n+A+1)!}{(n+1)!(n+1)!}, \quad A \leqslant n.$$

Eqn. (II.4-63) is linear, in contrast to eqn. (II.3-36). Therefore, it can be solved exactly in terms of the LLog coefficients of t_0^0-partial wave for the $\pi\pi$-scattering, which will be shown in the next chapter. The first few values of v_n (at $N = 3$) obtained by the numerical computation are

$$v_n = \left\{1, 1, \frac{43}{36}, \frac{143}{108}, \frac{15283}{9720}, \frac{2578307}{1458000}, \frac{888770227}{428652000}, \frac{26311049231}{11202105600}, ..\right\}.$$

This values coincide with the known results of the direct 2-loop calculation in ChPT [48], and the four-loop indirect LLog compilation of ref. [22].

One can solve eqn. (II.4-63) in the large-N limit. Substituting expansion (II.3-53) into eqn. (II.4-63) and finding the solution in the form

$$v_n = \sum_{k=0}^{n} \left(\frac{N}{2}\right)^{n-k} v_n^{(k)}, \tag{II.4-65}$$

we find that the leading order large-N is

$$v_n^{(0)} = 1.$$

The next-to-leading order is expressed in terms of $\omega_{n0}^{(1)}$

$$v_n^{(1)} = \sum_{k=1}^{n} \frac{1}{k}\left[\Omega_k^{00} + \omega_{k,0}^{(1)}\right] = \sum_{k=1}^{n} \left(\frac{2(n+1)}{k(k+1)} - \frac{1}{k}\right)\left(\Omega_k^{00} + \frac{2}{k-1}\sum_{i=1}^{k-1}\left(\frac{i!(k-i)!}{(k+1)!}\right)^2\right),$$

where for the last equality we have used exp. (II.3-56). The $n \to \infty$ asymptotic behavior of $v_n^{(1)}$ can be found analytically, it reads

$$v_n^{(1)}\Big|_{n\to\infty} \sim n(4 - \frac{\pi^2}{6} - 4\ln 2) + (3 - \frac{\pi^2}{6} - 2\ln 2) + \mathcal{O}(n^{-1}).$$

So we conclude that the large-N expansion works very bad for the $O(N+1)/O(N)$ σ-model, because starting from some n (for scalar FF it is $n \sim 1.13N$) the next-to-leading large-N order outruns the leading order and keeps growing. Moreover, one can show that $v_n^{(k)}$ grows as n^k at large n. Therefore, the higher orders of the large-

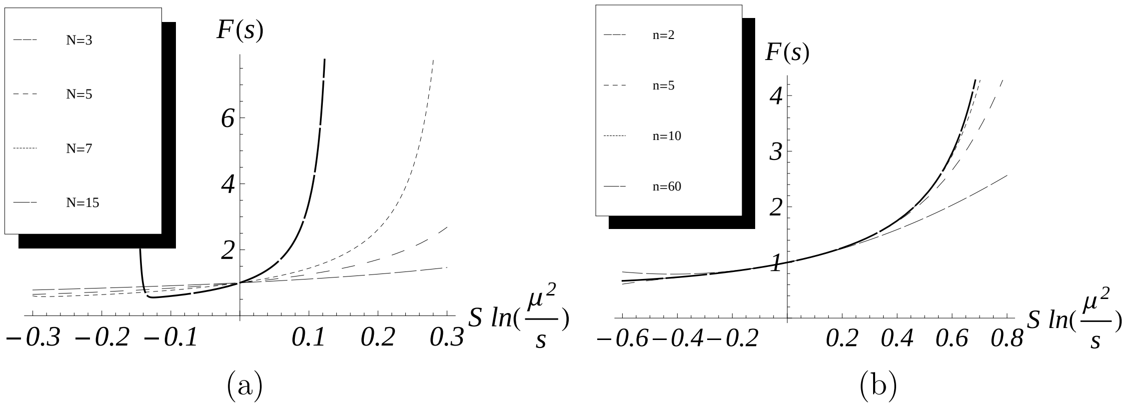

Figure II-7: (a)The plot of the scalar form factor with different number of pions. (b)The plot of the first n terms of the scalar form factor (II.4-59). Both plots normalized as $F(0) = 1$.

N expansion dominate over the leading order. This is demonstrated in fig.II.7(a), the curves with different values N strongly differ from each over. Also the higher order large-N series diverges if one sums them independently, whereas the complete answer converge perfectly. One can see that in fig.II.7(b): the first 10 terms give approximately the full result.

Vector form factor The vector or electro-magnetic form factor of the pion is also a well investigated object. The consideration of its LLog contribution is very similar with the scalar FF case. The definition of the vector FF is

$$\langle 0|J^\mu(0)|\pi^a(p)\pi^b(p')\rangle \;=\; i\varepsilon^{3ab}(p'-p)^\mu \mathbb{F}(s)\ . \tag{II.4-66}$$

The definition of the electro-magnetic current J_μ at all orders can be done in the following way

$$J_\mu = \sum_{n=0}^{\infty} f_n \varepsilon_{3ab}\ \ \partial^{2n}\left(\pi_a\partial^\mu\pi_b\right). \tag{II.4-67}$$

One can find that this vertex can be obtained formally from V_{n1} by taking out the left pair of field. Thus, its anomalous dimension is closely connected with $\beta(i,A;n-i,1/1)$. Calculating of the diagram in fig.II.6 with vector current (Appendix A.3), one finds that the one-loop anomalous dimension of the vector current (II.4-67) is

$$Z_V^{n,i,A} \;=\; \frac{1}{3}\Omega^{A1}_{n-i}\ . \tag{II.4-68}$$

Substituting Z_V into eqn. (II.4-63) one obtains the LLog coefficients for $\mathbb{F}(s)$. The first few of them are

$$\left\{1, -\frac{1}{6}, \frac{1}{72}, -\frac{91}{1296}, \frac{3607}{155520}, -\frac{7124897}{163296000}, \frac{937784623}{41150592000}, -\frac{134135230877}{4032758016000}, \frac{189853887100991}{8710757314560000}\right\}.$$

The first tree numbers are known from the explicit two-loop calculation [48]. Note, that the consequence of LLog coefficients for the vector FF is sign-variable.

The anomalous dimension of the vector current (II.4-68) does not contain N, and it leads to the simplification of the large-N expansion evaluating. The leading and next-to-leading terms of the large-N expansion for the LLog coefficient of vector FF are

$$\mathbb{V}_n = \delta_{0,n}\left(\frac{N}{2}\right)^n + \left(\frac{N}{2}\right)^{n-1}\frac{(-1)^{n-1}}{(n+1)(n+2)} + \mathcal{O}(N^{n-2}).$$

One can find that the k-th order of large-N expansion at large n behaves as $\mathbb{V}_n^{(k)} \sim n^{k-3}$. Therefore, although the large-N expansion works much better for the vector FF than for the scalar FF, it also fails at the $n \sim N^3$ order of the expansion.

Tensor form factor The tensor or the gravitational form factor of the pion is a more complicated object from the technical point of view . From the physical point of view the tensor FF often appears in hard processes with the participation of pions. The tensor FF has two components

$$\langle 0|J^{\mu\nu}(0)|\pi^a(p)\pi^b(p')\rangle \;=\; \delta^{ab}\Big[\frac{1}{2}(g_{\mu\nu}q^2 - q_\mu q_\nu)\theta_1(s) + \frac{1}{2}P_\mu P_\nu \theta_2(s)\Big]; \qquad \text{(II.4-69)}$$

where

$$s = (p'+p)^2\ , \quad q_\mu = (p+p')_\mu\ , \quad P_\mu = (p'-p)_\mu\ .$$

The lowest order of the momentum-energy tensor $J_{\mu\nu}$ is given by the variation of the Lagrangian (II.2-5) over the metric $g_{\mu\nu}$,

$$J^{\mu\nu}(0) \;=\; \partial_\mu\pi^a\partial_\nu\pi^a - \frac{g_{\mu\nu}}{2}\partial_\alpha\pi^a\partial_\alpha\pi^a + \mathcal{O}(\pi^4), \qquad \text{(II.4-70)}$$

where the second term results from the variation of the invariant integral measure, $\sqrt{g}$, in the action. Comparing exp. (II.4-70) with definition (II.4-69), one finds that both tensor FFs are normalized on unity, $\theta_{1,2}(s) = 1 + \mathcal{O}(s)$.

All order tensor sources can be chosen in the following way

$$J_1^{\mu\nu}(0) \;=\; \sum_{n=0}^{\infty} f_n\partial^{2n}\Big(\partial_\mu\pi^a\partial_\nu\pi^a - \frac{g_{\mu\nu}}{2}\partial_\alpha\pi^a\partial_\alpha\pi^a\Big)\ , \quad f_0 = 1 \qquad \text{(II.4-71)}$$

However, the renormalization of this source demands the introduction of another tensor current which does not exist at the lowest level:

$$J_2^{\mu\nu}(0) \;=\; \sum_{n=0}^{\infty} \frac{h_n}{4} \partial^{2n}\left(\pi^a \nabla_\mu \nabla_\nu \pi^a\right) \,, \quad h_0 = 0. \tag{II.4-72}$$

The renormalization of these two currents is not diagonal, i.e. the constants f and h take apart in the renormalization of each other. Therefore, the anomalous dimensions can be written as a matrix

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} f_n \\ h_n \end{pmatrix} \;=\; \sum_{i=0}^{n-1} \sum_{A=0}^{i} \begin{pmatrix} Z^{ff} & Z^{fh} \\ Z^{hf} & Z^{hh} \end{pmatrix} \begin{pmatrix} f_n \\ h_n \end{pmatrix} g_{iA} + \mathcal{O}(fg^{\langle 2 \rangle}). \tag{II.4-73}$$

Defining the LLog coefficient of the evolution of the currents J_1 and J_2 similar to the definition (II.4-62)

$$\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = \frac{(g_{10})^{-n}}{f_0} \frac{\hat{H}_1^n}{n!} \begin{pmatrix} f_n \\ h_n \end{pmatrix},$$

we obtain the recursive equations:

$$\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{A=0}^{n-i} \begin{pmatrix} Z^{ff} & Z^{fh} \\ Z^{hf} & Z^{hh} \end{pmatrix} \begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix} \omega_{n-i,A}. \tag{II.4-74}$$

The initial condition for the equation $\binom{\phi_0}{\psi_0} = \binom{1}{0}$.

The tree expression for the currents gives

$$\langle 0 | J_1^{\mu\nu}(0) + J_2^{\mu\nu}(0) | \pi^a(p) \pi^b(p') \rangle_{\text{tree}} \;=\; \sum_{n=0}^{\infty} \frac{s^n}{2} \Big[g_{\mu\nu} q^2 - q_\mu q_\nu + P_\mu P_\nu \Big] f_n - P_\mu P_\nu h_n. \tag{II.4-75}$$

Thus, the tensor FF LLog coefficients come from the combination of the LLog coefficients for the currents in the following way:

$$\begin{pmatrix} \theta_1(s) \\ \theta_2(s) \end{pmatrix} = \sum_{n=0}^{\infty} \hat{S}^n \ln^n \left(\frac{\mu^2}{-s} \right) \begin{pmatrix} \phi_n \\ \phi_n - \psi_n \end{pmatrix}. \tag{II.4-76}$$

The matrix of the anomalous dimensions is obtained by the direct calculation:

$$\begin{pmatrix} Z^{ff} & Z^{fh} \\ Z^{hf} & Z^{hh} \end{pmatrix} = \sum_{B=0}^{n-i} \left(N\delta^{AB} + 2\Omega^{AB}_{n-i} \right) \left[\frac{\delta_{B0}}{3} \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} + \frac{\delta_{B2}}{15} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix} \right]. \tag{II.4-77}$$

The numerical calculation of the first few elements at $N = 3$ gave

$$\theta_1 : \left\{ 1, \frac{2}{3}, \frac{16}{27}, \frac{77}{162}, \frac{10111}{19440}, \frac{391051}{850500}, \frac{1515484571}{2893401000}, \frac{4184563757183}{8506598940000} \right\},$$

$$\theta_2 : \left\{1, 0, -\frac{1}{54}, 0, -\frac{337}{19440}, \frac{6407}{1913625}, -\frac{167062219}{11573604000}, \frac{57017479021}{11342131920000}\right\},$$

which is can be checked only by a 1-loop calculation presented in ref. [49]. Note, that the θ_2 LLog coefficients at 1 and 3 loop order are exactly zero, somehow it indicates that for $\theta_2(s)$ the NLLog part gives a stronger contribution.

The system of recursive equations (II.4-74) can be diagonalized. The operators for the currents $J_{1,2}$ can be combined into the operators with only the zeroth and the second partial waves: $2J_1 - J_2 \sim P_0$ and $J_1 - J_2 \sim P_2$. Due to the "diagonality" of the Legendre operator basis (II.3-45), these operators renormalize only them-self. Therefore, instead of the system of equations (II.4-74) one obtains two independent equations:

$$\begin{aligned}
(\phi-\psi)_n &= \frac{1}{n}\sum_{i=0}^{n-1}\sum_{A,B=0}^{n-i,i}\frac{\delta_{2B}}{15}\Big[N\delta^{AB}+2\Omega^{AB}_{n-i}\Big](\phi-\psi)_i\omega_{n-i,A}\ , \quad (\phi-\psi)_0=1,\\
(2\phi-\psi)_n &= \frac{1}{n}\sum_{i=0}^{n-1}\sum_{A,B=0}^{n-i,i}\frac{-\delta_{0B}}{6}\Big[N\delta^{AB}+2\Omega^{AB}_{n-i}\Big](2\phi-\psi)_i\omega_{n-i,A}\ , \quad (2\phi-\psi)_0=2.
\end{aligned}$$

The large-N investigation gives no new information from the methodological point of view. For completeness we provide the first two orders of large-N expansion.

$$(2\phi-\psi)_n = \sum_{k=0}^{n}\left(\frac{N}{2}\right)^{n-k}\varrho^{(k)}_n, \quad (\phi-\psi)_n = \sum_{k=0}^{n}\left(\frac{N}{2}\right)^{n-k}\rho^{(k)}_n$$

$$\begin{aligned}
\varrho^{(0)}_n &= \frac{2}{n!}\frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})},\\
\varrho^{(1)}_n &= \frac{1}{3n}\sum_{i=0}^{n-1}\varrho^{(1)}_i + \frac{1}{3n}\sum_{i=0}^{n-1}\Big[\Omega^{00}_{n-i}\varrho^{(0)}_i + \omega^{(1)}_{n-i,0}\varrho^{(0)}_i\Big]\ , \quad \varrho^{(1)}_0 = 0,\\
\rho^{(0)}_n &= \delta_{n,0}, \qquad \rho^{(1)}_n = \frac{-2}{15n}\Big[\Omega^{02}_n + \omega^{(1)}_{n2}\Big].
\end{aligned}$$

II.5 LLog for $\pi\pi$ scattering in other models

In this section we present the results of the calculation of the LLog coefficients in the massless ChPT and the matrix model. These models have more complicated structure than the $O(N+1)/O(N)$ σ-model (II.2-5).

Massless ChPT ChPT is an EFT for the low energy QCD, see section 1.2. In the massless limit the Lagrangian of ChPT is invariant under $SU(N)_L\times SU(N)_R/SU(N)_V$ transformations, where N is the number of quark flavors. The lowest order Lagrangian has the form (I.3-11):

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{tr}\left[\partial_\mu U \partial_\mu U^\dagger\right], \qquad \text{(II.5-78)}$$

where U is a matrix of Goldstone fields $U = \exp[i\frac{\phi^a\lambda^a}{F}]$, λ^a is a generator of $SU(N)$ group, and F is the lowest coupling. Since $SU(2) = O(3)$ the 2-flavour ChPT is equivalent to the Weinberg model (II.2-5).

For our estimation we have to represent the Lagrangian (II.5-78) in the form of (I.1-2). Expanding the field U over the fields π and taking the trace one obtains

$$\mathcal{L}_2 = -\frac{1}{2}\pi^a\partial^2\pi^a - \frac{1}{4F^2}\left[\frac{\delta^{ab}\delta^{cd}}{N} + \frac{1}{2}d^{ab\alpha}d^{\alpha cd}\right]\left(\pi^a\pi^b\partial^2\pi^c\pi^d\right) + \mathcal{O}(\pi^6), \qquad \text{(II.5-79)}$$

where $\{t^a, t^b\} = d^{abc}t^c$. The only difference from the previous case consist in the group coefficient of the vertex.

In the $SU(N)$-symmetric Lagrangian the two possible independent group constructions of the vertices can appear, namely $\sim \mathrm{tr}[UU^\dagger]$ and $\sim \mathrm{tr}[U]\mathrm{tr}[U\dagger]$. In the four-field part of the Lagrangian they results into $\left[\frac{\delta^{ab}\delta^{cd}}{N} + \frac{1}{2}d^{ab\alpha}d^{\alpha cd}\right]$ and $\delta^{ab}\delta^{cd}$ respectively. The second structure is not presented in the initial Lagrangian (II.5-79), but it appears at higher orders. This situation is similar to tensor FF consideration. We introduce two sets of operators with their own coupling constants: g_n and h_n. During the renormalization they mix with each other, therefore, the β-function coefficients compose the matrix.

Here, without annoying details we present the final result of our calculation. The $\pi\pi$ scattering amplitude in the model (II.5-78) at LLog accuracy has the form (the indices and momenta are in the same position as in fig.II.4):

$$\begin{aligned}\mathcal{M} &= (4\pi)^2\sum_{n=0}^{\infty}\hat{S}^n \ln^{n-1}\left(\frac{\mu^2}{|s|}\right)\cdot \qquad \text{(II.5-80)}\\ &\cdot\sum_{C=0}^{n}\left\{\left[\frac{\delta^{ab}\delta^{cd}}{N} + \frac{1}{2}d^{ab\alpha}d^{\alpha cd}\right]\omega_{nC} + \delta^{ab}\delta^{cd}\upsilon_{nC}\right\}P_C\left(1+\frac{2t}{s}\right)\\ &+\begin{pmatrix} b\leftrightarrow c\\ p_2\leftrightarrow p_3\end{pmatrix} + \begin{pmatrix} b\leftrightarrow d\\ p_2\leftrightarrow p_4\end{pmatrix}.\end{aligned}$$

The coefficients ω_{nC} and υ_{nC} satisfy the equations

$$\begin{aligned}\omega_{nC} &= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{A,B=0}^{i,n-i}\left[\beta^g_{gg}\omega_{iA}\omega_{n-i,B} + \beta^g_{gh}\omega_{iA}\upsilon_{n-i,B}\right] \qquad \text{(II.5-81)}\\ \upsilon_{nC} &= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{A,B=0}^{i,n-i}\left[\beta^h_{gg}\omega_{iA}\omega_{n-i,B} + \beta^h_{gh}\omega_{iA}\upsilon_{n-i,B} + \beta^h_{hh}\upsilon_{iA}\upsilon_{n-i,B}\right],\end{aligned}$$

where the β-functions obtained from the direct calculation of the diagrams of fig.II.3.

are

$$\begin{aligned}
\beta^g_{gg} &= R\Big[\frac{1}{2}\Big(\frac{N}{4}-\frac{1}{N}\Big)\frac{\delta_{AC}\delta_{BC}}{2C+1}+\Big(\frac{N}{8}-\frac{1}{N}\Big)\Big(\frac{\delta_{BC}\Omega^{AC}_i}{2C+1}+\frac{\delta_{AC}\Omega^{BC}_{n-i}}{2C+1}\Big)\\
&\quad+\Big(\frac{N}{8}-\frac{2}{N}\Big)\frac{\Omega^{AC}_i\Omega^{BC}_{n-i}}{2C+1}+\frac{N}{8}\sum_{J=0}^{n}\frac{(1-(-1)^J)\Omega^{AJ}_i\Omega^{BJ}_{n-i}\Omega^{JC}_n}{2J+1}\Big]\\
\beta^g_{gh} &= R\Big[\frac{\delta_{AC}\Omega^{BC}_{n-i}}{2C+1}+2\sum_{J=0}^{n}\frac{\Omega^{AJ}_i\Omega^{BJ}_{n-i}\Omega^{JC}_n}{2J+1}\Big]
\end{aligned}$$

$$\begin{aligned}
\beta^h_{gg} &= R\Big[\Big(\frac{1}{8}+\frac{1}{4N^2}\Big)\frac{\delta_{AC}\delta_{BC}}{2C+1}+\Big(\frac{1}{8}+\frac{1}{2N^2}\Big)\Big(\frac{\delta_{BC}\Omega^{AC}_i}{2C+1}+\frac{\delta_{AC}\Omega^{BC}_{n-i}}{2C+1}\Big)\\
&\quad+\Big(\frac{1}{8}+\frac{1}{N^2}\Big)\frac{\Omega^{AC}_i\Omega^{BC}_{n-i}}{2C+1}+\frac{1}{8}\sum_{J=0}^{n}\frac{(1+(-1)^J)\Omega^{AJ}_i\Omega^{BJ}_{n-i}\Omega^{JC}_n}{2J+1}\Big]\\
\beta^h_{gh} &= R\Big[\Big(\frac{N}{2}-\frac{1}{2N}\Big)\frac{\delta_{AC}\delta_{BC}}{2C+1}+\Big(\frac{N}{2}-\frac{1}{N}\Big)\frac{\delta_{BC}\Omega^{AC}_i}{2C+1}\Big]\\
\beta^h_{hh} &= R\Big[\frac{N^2-1}{4}\frac{\delta_{AC}\delta_{BC}}{2C+1}+\frac{1}{2}\frac{\delta_{BC}\Omega^{AC}_i}{2C+1}+\frac{1}{2}\frac{\delta_{AC}\Omega^{BC}_{n-i}}{2C+1}+\sum_{J=0}^{n}\frac{\Omega^{AJ}_i\Omega^{BJ}_{n-i}\Omega^{JC}_n}{2J+1}\Big]
\end{aligned}$$

where the coefficient R is the parity factor

$$R=\frac{1+(-1)^C}{2}(1+(-1)^A)(1+(-1)^B).$$

The initial conditions are $\omega_{10}=1$ and $v_{10}=0$.

At $N=2$, the structure constants are $d^{abc}=0$. Therefore, the LLog coefficients in exp. (II.5-80) are (ω_n+v_n). The system of equation (II.5-81) can be diagonalized, and the equation for (ω_n+v_n) will contain only β^{hh}_h. Thus, LLog coefficients of the model coincide with ones obtained for the $O(4)/O(3)$ σ-model, see (II.3-43,II.3-44,II.3-52).

In table II.3 we presented the first few values of the coefficients. The coefficients ω are polymons in N with maximal power $(n-1)$. The polinomiality in N of the coefficients also follows from the usual rules of N-calculations. The cancelation of the inverse powers of N, which are presented in the equations (II.5-81) is a strong check of our calculation. Also one can see that all v_{nC} are subleading at large-N limit. Consequently the pure δ-index-structures should be subleading in the rules for large-N calculations for $SU(N)$ groups.

The equation for ω's is very good object for the investigation of the large-N limit of non trivial theories, because this equation, in contrast to the usual RGE for renormalizable theories, "fills" the diagram structure at all orders of perturbation expansion much better. The only known rule for the large-N expansion at $SU(N)$ groups is that the leading order terms are contained only in the planar graphs [51].

Table II-3: The first few values of LLog coefficients ω_{nC} and υ_{nC} for $\pi\pi$ scattering in massless N-flavored ChPT

$n \setminus C$	0	2	4
1	1	0	0
2	$\frac{5N}{72}$	$\frac{N}{72}$	0
3	$\frac{5N^2}{144}+\frac{25}{48}$	$\frac{5}{48}$	0
4	$\frac{547N^3}{77760}-\frac{403N}{14400}$	$\frac{79N}{5040}-\frac{377N^3}{1088640}$	$\frac{N^3}{40320}+\frac{713N}{302400}$
5	$\frac{11509N^4}{5832000}+\frac{101339N^2}{2592000}+\frac{42427}{172800}$	$-\frac{27023N^4}{522547200}+\frac{16661N^2}{2073600}+\frac{3071}{48384}$	$\dots$

$n \setminus C$	0	2	4
1	0	0	0
2	$\frac{5}{24}$	$\frac{1}{24}$	0
3	$\frac{85N}{576}$	$\frac{5N}{576}$	0
4	$\frac{3949N^2}{43200}+\frac{1463}{14400}$	$\frac{191N^2}{120960}+\frac{209}{5040}$	$\frac{13N^2}{201600}+\frac{209}{100800}$
5	$\frac{1990271N^3}{37324800}+\frac{14749N}{230400}$	$\frac{4931N^3}{20901888}+\frac{109N}{8064}$	$\frac{4553N^3}{174182400}+\frac{851N}{537600}$

But planar graphs also contain subleading terms in large-N parts. The LLog terms are also present in the planar graphs only. Therefore, investigating eqns. (II.5-81) can give us a key for understanding the rules of large-N counting behind the planar graphs.

The large-N behavior of the coefficients ω is described by eqn. (II.3-36) with the β-function

$$\beta^{SU(N)}_{\text{Large-N}} = R\frac{N}{8}\Big[\frac{\delta_{AC}\delta_{BC}}{2C+1}+\Big(\frac{\delta_{BC}\Omega_i^{AC}}{2C+1}+\frac{\delta_{AC}\Omega_{n-i}^{BC}}{2C+1}\Big)+\frac{\Omega_i^{AC}\Omega_{n-i}^{BC}}{2C+1} + \sum_{J=0}^{n}\frac{(1-(-1)^J)\Omega_i^{AJ}\Omega_{n-i}^{BJ}\Omega_n^{JC}}{2J+1}\Big]. \qquad \text{(II.5-82)}$$

The equation for the coefficient υ can be dropped out because β^h_{gg} is subleading in the large-N counting. We see that the theory, which describes the large-N limit of $SU(N)$ theories, is not so simple as for $O(N)$-model. Unfortunately, the solution of this equation is unknown.

The matrix model The matrix models are very popular EFTs since they provide wide possibilities to realize the difficult enclosed symmetries. Here we consider the simplest matrix extension of the $O(N)$ model (II.2-5), namely the $O(N+K)/O(K)\times O(N)$ σ-model. It is given by the initial Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a_\alpha)(\partial_\mu\phi^a_\alpha) - \frac{1}{2F^2}(\phi^a_\alpha\partial_\mu\phi^b_\alpha)(\phi^a_\beta\partial_\mu\phi^b_\beta)\ , \qquad \text{(II.5-83)}$$

where the lower indices run from 1 to $(N+K)$ and the upper indices run from 1 to K. The condition that the fields lie on the corresponding topological surface is

$$\sum_{\alpha=1}^{N} \phi_\alpha^a \phi_\alpha^b \;=\; \delta^{ab} F^2. \tag{II.5-84}$$

In order to present the Lagrangian (II.5-83) in the form (I.1-2) one has to integrate out the degrees of freedom fixed by relations (II.5-84). Let us choose as dependent degrees of freedom the upper components of ϕ : $\phi_\alpha^a = (U_1^a, .., U_K^a, \pi_{K+1}^a, .., \pi_{K+N}^a)$. The U-fields can be expressed in terms of π-fields with the help of the (II.5-84): $U_\alpha^a = \delta^{a\alpha} F - \frac{1}{2F}\pi_\beta^a \pi_\beta^\alpha + \mathcal{O}(\pi^4)$. The fields π are $K \times N$ matrices. The Lagrangian (II.5-83) in terms of fields π has the form

$$\mathcal{L}_2 = -\frac{1}{2}\pi_\alpha^a \partial^2 \pi_\alpha^a + \frac{1}{4F^2}\pi_\alpha^a \pi_\beta^b \partial^2 \pi_\beta^a \pi_\alpha^b + \mathcal{O}(\pi^6).$$

The further consideration repeats all previous cases. Therefore, we give only the final result of our calculation. The $\pi\pi$ scattering at the LLog approximation has the following structure:

$$\begin{aligned}\mathcal{M} \;=\; & (4\pi)^2 \sum_{n=0}^{\infty} \hat{S}^n \ln^{n-1}\left(\frac{\mu^2}{|s|}\right) \cdot \\ & \cdot \sum_{C=0}^{n} \Big\{ \left[\delta^{a_1a_3}\delta^{a_2a_4}\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} + (-1)^C \delta^{a_1a_4}\delta^{a_2a_3}\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\right]\omega_{nC} \\ & + \delta^{a_1a_2}\delta^{a_3a_4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} v_{nC} \Big\} P_C\left(1+\frac{2t}{s}\right) + (2\leftrightarrow 3) + (2\leftrightarrow 4),\end{aligned} \tag{II.5-85}$$

where the upper indices running from 1 to K are denoted by a_i, the lower indices running from 1 to N are denoted by α_i, the enumerating of the pions fields is coincided with enumerating of its momenta, see fig.II.4.

The coefficients ω and v satisfy the system of equations (II.5-81) with β-functions:

$$\begin{aligned}\beta_{gg}^g \;=\; & ((-1)^A + (-1)^{B+C})\frac{\Omega_{n-i}^{BA}\Omega_n^{AC}}{2A+1} + ((-1)^B + (-1)^{A+C})\frac{\Omega_i^{AB}\Omega_n^{BC}}{2B+1} \\ & + (K + (-1)^{A+B+C}N)\sum_{J=0}^{n}\frac{(-1)^J \Omega_i^{AJ}\Omega_{n-i}^{BJ}\Omega_n^{JC}}{2J+1}, \\ \beta_{gh}^g \;=\; & (1+(-1)^B)\Big[\frac{\delta_{AC}\Omega_{n-i}^{BC}}{2C+1} + (1+(-1)^{A+C})\sum_{J=0}^{n}\frac{\Omega_i^{AJ}\Omega_{n-i}^{BJ}\Omega_n^{JC}}{2J+1}\Big],\end{aligned}$$

$$
\begin{aligned}
\beta^h_{gg} &= (1+(-1)^C)\Big[((-1)^A+(-1)^B)\frac{\Omega_i^{AC}\Omega_{n-i}^{BC}}{2C+1}+\frac{\delta_{AB}\Omega_n^{AC}}{2C+1}\Big],\\
\beta^h_{gh} &= (1+(-1)^B)\Big[\frac{\delta_{AC}\delta_{BC}}{2C+1}+((-1)^A N+K)\frac{\delta_{BC}\Omega_i^{AC}}{2C+1}\Big],\\
\beta^h_{hh} &= \frac{(1+(-1)^A)(1+(-1)^B)}{16}\Big[NK\frac{\delta_{AC}\delta_{BC}}{2C+1}+2\frac{\delta_{AC}\Omega_{n-i}^{BC}}{2C+1}+2\frac{\delta_{BC}\Omega_i^{AC}}{2C+1}\\
&\qquad +2(1+(-1)^C)\sum_{J=0}^{n}\frac{\Omega_i^{AJ}\Omega_{n-i}^{BJ}\Omega_n^{JC}}{2J+1}\Big].
\end{aligned}
$$

We do not present here the explicit expressions for ω and v, because they are quite large. The theory (II.5-83) is symmetrical under interchange N and K. Using the condition (II.5-84) we had broken this symmetry and, as a result, β-functions has no $N \leftrightarrow K$ symmetry. But the coefficients ω and v are symmetric under interchange of N and K, that confirms our calculation. Also one can check that in the limit N(or K)$\to 1$ the model (II.5-83) coincides with the model (II.2-5).

We also note that in this model all components (odd and even) of the partial wave basis are presented, in contrast with both previously considered cases? where were only even partial waves. It is a consequents of a non trivial $(t \leftrightarrow u)$ crossing symmetry, which is also seen in the expression for $\mathcal{M}$ (II.5-85). However, for consideration of this model one can use the Legendre polynomial operator (II.3-41) without any addition extensions.

II.6 LLogs for 6π amplitude

In this section in order to demonstrate the power of the presented method we derive the recursive equations for the LLog coefficients of the 6-pion amplitude . The calculation of kernels of the equations was not done. However, we suppose that the derivation of the corresponding equations is a good example of the stated above method. We consider a model with leading ϕ^4 interaction like massless ChPT or the $O(N+1)/O(N)$ model. The concrete form of the model influences only on the expression of β-function.

As in case of the $\pi\pi$ scattering, one has to write down the most general perturbative expansion for the 6π amplitude. The g-structure of the 6π amplitude coincides with the g-structure of $\beta^{(6)}$-function (II.2-15) with addition of the 0-loop part:

$$
\begin{aligned}
\mathcal{M}^{(6)} &= \langle \pi^2|S-I|\pi^4\rangle \qquad \text{(II.6-86)}\\
&= \sum_{n=1}^{\infty} E^{2n}\sum_{l=0}^{n-1}\sum_{j=0}^{l+1}\sum_{p}|g^{\langle l+p-j+2\rangle}|_{n-j+1}\sum_{m=0}^{l-\max[j-1,0]}\ln^m(\mu^2)f_{n,j,l,p}(k),
\end{aligned}
$$

where k are all momenta variables of the 6π amplitude.

Differentiating the coefficient of E^{2n} in $\mathcal{M}^{(6)}$ $(n-1)$ times one obtains the LLog

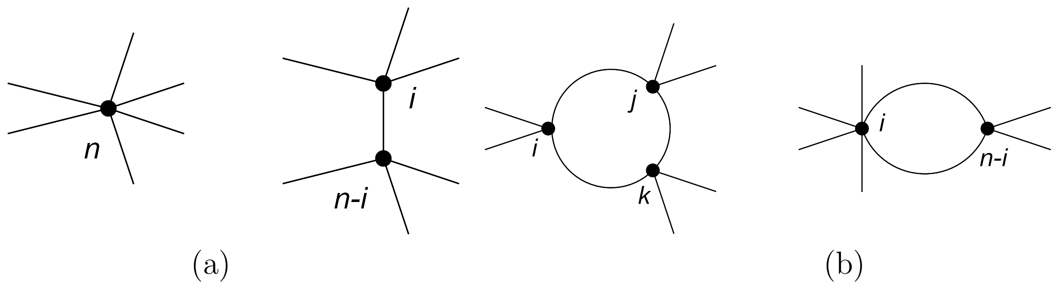

Figure II-8: (a) Tree diagrams for the $\pi\pi \to \pi\pi\pi\pi$ amplitude (the crossed diagrams should be added). (b) The one-loop diagrams contributed to the $\beta_n^{(6)}$.

coefficient:

$$\left[\mu^2 \frac{\partial}{\partial \mu^2}\right]^{n-1} \mathcal{M}^{(6)}\big|_{E^{2n}} = (n-1)! \left[f_n(k)|g^{\langle n+1\rangle}|_{n+1} + \tilde{f}_n(k)|g^{\langle n\rangle}|_n\right]. \quad \text{(II.6-87)}$$

On the one hand, the momentum dimension of the expression is $(-2n-2)$. On the other hand, exp. (II.6-87) contains only the first order coupling constants, as it follows form property (II.2-18). Therefore, the second term in eqn. (II.6-87) contains one $g_1^{(6)}$ vertex. However, usually, due to the symmetries of Lagrangian $g_1^{(6)} \sim (g_1)^2$, for example (II.2-5). Expanding eqn. (II.6-87) over complete basis P_C in the momentum and group spaces one finds:

$$\left[\mu^2 \frac{\partial}{\partial \mu^2}\right]^{n-1} \mathcal{M}^{(6)}\big|_{E^{2n} P_C} = (n-1)! R_{nC} g_{10}^{n+1}, \quad \text{(II.6-88)}$$

where R_{nC} is the required LLog coefficient.

The differentiation over μ^2 is equivalent to the action of the operator $\hat{H}$. Acting by $\hat{H}$ $(n-1)$ times and performing the analysis of the summation limits one obtains that only two terms of the sum (II.3-29) give a non-zero contribution and that only one-loop part of $\hat{H}$ is presented:

$$\hat{H}^{n-1} \mathcal{M}^{(6)}\big|_{E^{2n}} = \hat{H}_1^{n-1} \left[f_0(k)|g^{\langle 2\rangle}|_{n+1} + \tilde{f}_0(k)|g^{\langle 1\rangle}|_n\right]. \quad \text{(II.6-89)}$$

The term in the square brackets is the tree order of the 6π-amplitude, which contain two graphs. One of them is a pure $V_n^{(6)}$-vertex. Another is the tree with two $V^{(4)}$ vertices, see fig.II.8(a). Expanding exp. (II.6-89) over the basis P_C one obtains

$$\hat{H}^{n-1} \mathcal{M}^{(6)}\big|_{E^{2n} P_C} = \hat{H}_1^{n-1} \left[\sum_{i=1}^{n} f(i, A; n+1-i, B/C) g_{iA}^{(4)} g_{n+1-i,B}^{(4)} + g_{nC}^{(6)}\right], \quad \text{(II.6-90)}$$

where $f(i, A; n+1-i, B/C)$ are known coefficients. We assume that the normalization of basis P_C is such that $\tilde{f}_0 = \sum_C P_C$.

We define the LLog coefficient of the $g^{(6)}$ evolution

$$æ_{nC} = \frac{\left(g^{(4)}_{10}\right)^{-n}}{(n-1)!}\hat{H}_1^{n-1} g^{(6)}_{nC}, \tag{II.6-91}$$

in complete analogy of definition (II.3-34). Expressions (II.6-88) and (II.6-90) are equal to each other, therefore, after simple reorganization one obtains that the LLog coefficient is

$$R_{nC} = \sum_{i=1}^{n} f(i, A; n+1-i, B/C)\omega_{iA}\omega_{n+1-i,B} + æ_{nC}. \tag{II.6-92}$$

The values of the coefficients ω_{nC} are fixed by eqn. (II.3-36). The equation for the coefficients $æ_{nC}$ can be obtained in the similar way. The part of operator $\hat{H}_1$, which acts on the $g^{(6)}$ coupling has the form

$$\hat{H}_1 \sim \left[\sum_{i+j+k=n} \beta_{i,j,k} g^{(4)}_i g^{(4)}_j g^{(4)}_k + \sum_{i=1}^{n-1} \tilde{\beta}_{i,n-i} g^{(6)}_i g^{(4)}_{n-i}\right] \frac{\partial}{\partial g^{(6)}_n},$$

where the auxiliary indices are omitted for compactness. The β-functions are given by the pole parts of the one-loop diagrams shown in fig. II.8(b). With the help of this expression, and definitions (II.3-34,II.6-91) one obtains a linear recursive equation for $æ_n$

$$æ_n = \frac{1}{n-1} \sum_{i+j+k=n} \beta_{i,j,k}\omega_i\omega_j\omega_k + \frac{1}{n-1}\sum_{i=1}^{n-1} \tilde{\beta}_{i,n-i} æ_i \omega_{n-i}. \tag{II.6-93}$$

The next step is the construction of the higher order Lagrangian and the calculation of the β-function. Note, that the higher order Lagrangian has to be built preserving the global symmetry of the theory. This gives some set of relations between the $g^{(6)}$ couplings and $g^{(4)}$ couplings, which would reflect on the β-functions. This calculating part of the consideration has not been done yet, since it has little practical interest.

There are also some hints which say that eqn. (II.6-93) can be greatly simplified, if one works in terms of the LLogs coefficients R_{nC}. It seems that some intrinsic symmetries of the β-function should exist, and the equation written in terms of R_n would be of the form of the FF equation, i.e. $R_n \sim \sum R_i\omega_{n-i}$. In the next chapter we will discuss the source of these hints.

II.7 Discussion

The main result of this chapter is the proof of the general relation (II.2-25). Together with the renorminvariance principle it gives formal definitions for the LLog coefficients of amplitudes. The formal definition of the LLog coefficients allows one to find the coefficients recursively. Such approach can be used in any EFT irrespective to the presence of masses. The masslessness of a theory restricts the number of the counterterms and therefore, the equations on LLog coefficients are closed. In the presence of masses the system n the LLog coefficients contains an infinite number of equations, because the renormalization of all possible types of vertices has to be taken into account.

Among all massless EFTs, the theories with the four-field leading interaction take a special place, because LLog terms in such theories are mostly made up by the renormalization of the four-field operators. On the diagrammatical level, it means that four-vertices are presented in a diagram without any restrictions, see discussion in the section II.1. Therefore, the leading logarithmal running of the couplings $g^{(4)}$, which is expressed with the help of coefficients ω_n (II.3-34), depends only on its own properties. Thus, the recursive equation (II.3-36) is closed, but non-linear. The recursive equations on the LLog coefficients of the other objects in the theory are linear, but contain the coefficients ω's or other senior quantities, for example see equations (II.4-63) and (II.6-93).

It can be easily understood through the consideration of the topological properties of the graphs. The operator $\hat{H}_1$ pull apart a vertex and inserts instead of it a loop of the same dimension. Thus, the form of the LLog recursive equation for some amplitude repeats the form of the one-loop graphs, which gave corrections to this amplitude. Then one can see the form correlation between the equations (II.3-36,II.4-63,II.6-93) and figs.II.3,II.6,II.8b, respectively. Therefore, obviously the graphs with four external lines only can reproduce themself, at one-loop level and without tadpole subgraphs.

A slightly different situation is in the EFTs of the ϕ^3-type. The general discussion and the properties of operator $\hat{H}$ remains unchanged. But the topology of the graphs in such a theory is completely different. The one-loop correction to the 3-point Green function is given by the graphs shown in fig.II.9a. The graphs contain the 2- and 4- vertices as well as 3-vertices. The one-loop graphs for the 2- and 3- point Green functions (fig.II.9c and fig.II.9b, respectively) also contain 2-,3-,4-vertices. Therefore the equations on the LLog coefficients for the 2-,3- and 4- point Green functions are

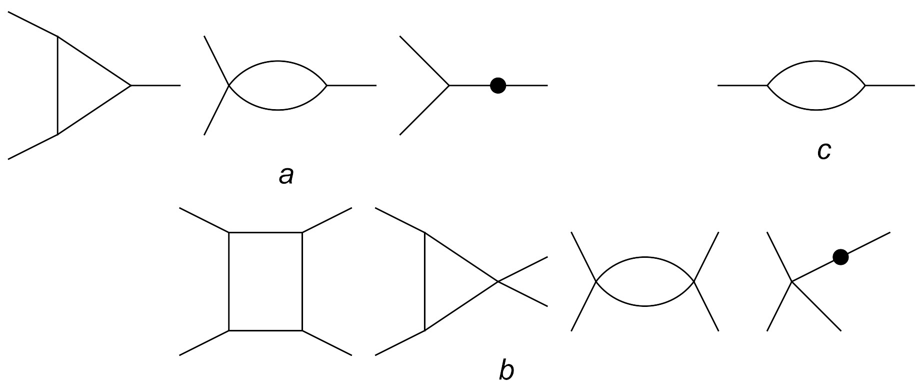

Figure II-9: Types of diagrams needed to calculate for obtaining the recursive equation for senior LLog coefficients in ϕ^4-type theories.

interlaced into the system of recursive equations

$$
\begin{aligned}
\eta_n &\sim \sum_{i+j+k=n} \eta_i\eta_j\eta_k + \sum_{i+j=n} \eta_i\omega_j + \sum_{i+j=n} \eta_i\gamma_j, \\
\omega_n &\sim \sum_{i+j+k+m=n} \eta_i\eta_j\eta_k\eta_m + \sum_{i+j+k=n} \eta_i\eta_j\omega_k + \sum_{i+j=n} \omega_i\omega_j + \sum_{i+j=n} \omega_i\gamma_j, \\
\gamma_n &\sim \sum_{i+j=n} \eta_i\eta_j, \\
& \eta_n \simeq \hat{H}_1^{n-1} g_n^{(3)}, \quad \omega_n \simeq \hat{H}_1^{n-1} g_n^{(4)}, \quad \gamma_n \simeq \hat{H}_1^{n-1} g_n^{(2)}.
\end{aligned}
$$

The junior quantities (like FF, 5-point Green function and so on) will be expressed through these three coefficients linearly.

The theories with space-time dimensions $D > 4$ can be considered in the same way. The equations which described the LLog behavior are the same. The only difference in the calculation is the contraction of the higher order Lagrangian and the expression for the β-functions. In the next chapter, we will investigate theories in $D > 4$ space-time dimensions in details.

Since the recursive equations on the LLog coefficients are universal and depend only on the topology of graphs, they are also correct for the renormalizable theories. The **only** difference between the renormalizable and non-renormalizable theories is the form of the equation kernel. For the renormalizable theories the kernel does not depend on n. For example, the recursive equation for the LLog coefficient of 4-point amplitude in usual ϕ^4-model is

$$\omega_n = \frac{9/2}{n-1} \sum_{i=1}^{n-1} \omega_i\omega_{n-i}, \qquad \text{(II.7-94)}$$

where $\frac{9}{2}$ is the well-known β -function coefficient in this model [81]. The solution of eqn. (II.7-94) is $\omega_n = (9/2)^{n-1}$, which gives the well-known one-loop logarithmical running of coupling:

$$g(\mu^2) = \sum_{n=1}^{\infty} \omega_n g^n(\mu_0^2) \ln\left(\frac{\mu^2}{\mu_0^2}\right) = \frac{g(\mu_0^2)}{1 - \frac{9}{2} g(\mu_0^2) \ln\left(\frac{\mu^2}{\mu_0^2}\right)}.$$

Note, that the kernel of the non-renormalizable equation (II.3-36) is also independent on n in the large-N limit (II.3-54). Therefore, the large-N limit for the $O(N)$ symmetric non-renormalizable EFT is effectively renormalizable, which in, e.g. [17],[50].

There is little literature on the non-linear recursive equations. One of the modern books on the properties of the non-linear recursive equations [52] is mainly devoted to the functional equations and their relations with recursive ones. With its help one can write an integral equation for the generating function for LLog coefficients (see Appendix C.2). Then the generation function for the coefficient ω satisfies the Hammerstein equation of the second kind. There are several known exact solutions for specific kernels, see e.g.[53]. In the next chapter we will discuss the possible ways to the solutions of the recursive equations.

The technique of the present chapter allows one to calculate the higher order of logarithms as well. There are two types of NLLogs. Some of them are "true" NLLogs, i.e. they come from the next-to-leading poles of the higher loop graphs, another are "false", i.e. they are LLogs of the subleading graphs (see the discussion after exp. (II.2-26)). The difference between these two types of logarithms is huge: for the "true" NLLogs one has to calculate the two-loop β-function; for the "false" NLLog the already calculated one-loop β-function is enough, but the values of the couplings of the second generation have to be fixed.

III

LLogs in EFT from analyticity and unitarity

The recursive equations for LLog coefficients in a massless theory can be derived using the analyticity of the amplitude, the unitarity relation, and the crossing symmetry. The LLogs terms are fixed by these three conditions, and the one-loop β-function has an universal form for ϕ^4 types of EFTs including pure renormalizable cases and theories in higher dimensions. The dependence on the explicit form of the Lagrangian is in the boundary conditions for recursive equation, which are given by the tree order.

The structure of the chapter is the following. First, we investigate the FFs in the $O(N{+}1)/O(N)$ σ-model using its analytical properties and obtain the recursive equation for FF LLog coefficients. After this we obtain the equation for the $\pi\pi$-scattering LLog coefficients in the EFT with an arbitrary global symmetry. We generalize the equation for the case of the mixed renormalizable and non-renormalizable interactions, as well as for the arbitrary even $D \geqslant 4$-dimension case. At the end of the chapter, we present examples of the calculation in several popular EFTs.

III.1 LLog for form factors

Let us consider the pion scalar form factor in the $O(N+1)/O(N)$ σ-model, (II.2-5). The formal definition of the scalar FF is

$$\langle 0|J(0)|\pi^a(p_1)\pi^b(p_2)\rangle = \delta^{ab}F(s), \qquad \text{(III.1-1)}$$

where $s=(p_1+p_2)^2$. The momentum expansion of this object was investigated in section II.2, and at the LLog approximation it reads

$$F(s) \quad = \quad 2B(4\pi)^2\sum_{n=0}^{\infty} v_n\hat{S}^n \ln^n\left(\frac{\mu^2}{-s}\right)+\mathcal{O}(\text{NLLog}), \qquad \text{(III.1-2)}$$

where the coefficient v_n satisfies eqn. (II.4-63) with the anomalous dimension (II.4-64).

The FF in the complex s-plane has singularities only on the right half ($s>0$) of the plane, see e.g.[54],[55]. This explains our choice of the logarithm argument in exp. (III.1-2). There are no one-particle intermediate states in the theories of the ϕ^4-type or at least they do not influence on the LLogs. Therefore, the only expected singularities are cuts, which appear due to the logarithms.

The cuts are subdivided into the two-,three- four-,etc. particle cuts, which have the branching points at $s=4m^2$, $9m^2$, $16m^2$, etc, respectively (for the system of the particles with equal masses m). In case of a massless theory, the branching points of the cuts glue together at $s=0$. However, one has to distinguish between the cuts since they contribute differently to the logarithm structure. Namely, the multi-particle cuts (the more-than-two-particle cuts) are a consequence of the non-leading logarithms. This statement can be proven as following.

Let us consider the discontinuity over the right cut ($s>0$) of the matrix element (III.1-1). According to the unitarity of the S-matrix one has

$$2\delta^{ab}\text{Disc}F(s)=\sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^3\vec{k}_i}{(2\pi)^2 2k_{0i}}\times \qquad \text{(III.1-3)}$$
$$\langle 0|J(0)|\pi^{a_1}(k_1)\pi^{a_2}(k_2)..\pi^{a_n}(k_n)\rangle\langle\pi^{a_1}(k_1)\pi^{a_2}(k_2)..\pi^{a_n}(k_n)|\pi^a(p_1)\pi^b(p_2)\rangle,$$

where $\text{Disc}F(s)\equiv\frac{1}{2}\left(\text{Im}F(s+i\varepsilon)-\text{Im}F(s-i\varepsilon)\right)$, the summation goes over all possible intermediate states starting from the two-pion state. The discontinuity decreases the power of the polynomial in $\ln(-s)$ by unity. Therefore, the left-hand side of eqn. (III.1-3) has the form

$$\text{Disc}F(s)\sim\underbrace{F^{-2}\sum_j F^{-2j}L^j}_{\text{LLogs}}+\underbrace{F^{-4}\sum_j F^{-2j}L^j}_{\text{NLLogs}}+\underbrace{F^{-6}\sum_j F^{-2j}L^j}_{\text{N}^2\text{LLogs}}+..\ ,$$

where all numerical coefficients are set to unity, $L=\ln(s)$ and F is the expansion parameter. We keep the powers of F, because they are our controllers of the expansion order. The amplitudes with different number of pions have a different counting of F-powers, namely

$$\langle 0|J(0)|(\pi)^k\rangle \sim \underbrace{F^{-k+2}\sum_j F^{-2j}L^j}_{\text{LLogs}} + \underbrace{F^{-k}\sum_j F^{-2j}L^j}_{\text{NLLogs}} + ..\ ,$$

$$\langle(\pi)^k|\pi\pi\rangle \sim \underbrace{F^{-k}\sum_j F^{-2j}L^j}_{\text{LLogs}} + \underbrace{F^{-k-2}\sum_j F^{-2j}L^j}_{\text{NLLogs}} + ..\ .$$

Comparing the right- and left-hand sides of eqn. (III.1-3) one finds that

$$\underbrace{F^{-2}\sum_j F^{-2j}L^j}_{\text{LLogs}} + \underbrace{F^{-4}\sum_j F^{-2j}L^j}_{\text{NLLogs}} + .. = \underbrace{F^{-2}\sum_j F^{-2j}L^j}_{k=2,\text{LLogs}} + \underbrace{F^{-4}\sum_j F^{-2j}L^j}_{\text{k=3,LLogs and k=2 NLLogs}} + ..\ . \qquad \text{(III.1-4)}$$

Therefore, the LLog coefficient on the left-hand side is related only to the two-particle cut. Moreover, only the LLog part of the $\pi\pi$-scattering amplitude can be taken into account. The next order of logarithms are composed of multi-particle intermediate states and NLLog parts of the lower states.

In the section II.1 we have considered the topological properties of the graphs contributing to the LLog in a massless theory, now we can add to this consideration the following observation. A graph can contain the LLog and, hence, the leading pole, if it is 2-particle reducible. Moreover, the k-particle reducible graphs can contain only N^{k-2}LLog or less.

Let us consider exp. (III.1-3). Our aim is to obtain the LLog coefficients of FF. Thus, we neglect all terms on the right-hand side of eqn. (III.1-3) except the term with $k = 2$. This term contains two factors. One of them is the scalar form factor (III.1-1) and the other is the matrix element of $\pi\pi$-scattering in isospin-0 channel. By definition, the $\pi\pi$ scattering matrix element is

$$\langle\pi^c(k_1)\pi^c(k_2)|\pi^a(p_1)\pi^b(p_2)\rangle = I + i(2\pi)^4\delta^{ab}\delta^{(4)}(k_1+k_2-p_1-p_2)T^0(s,t), \qquad \text{(III.1-5)}$$

where I is the identity, T^0 is the $\pi\pi$-scattering isospin-0 amplitude, s and t are usual Mandelshtam variables for a 4-point function. To be precise, in the right-hand side of exp. (III.1-3) we have $(S-I)$-matrix elements, whereas the definition (III.1-5) is given for the S-matrix element. The integration over δ-functions can be easily done in the center mass frame (c.m.f.) with $\vec{p} = -\vec{p}'$,

$$2\delta^{ab}\text{Disc}F(s) = \frac{i\delta^{ab}}{8(2\pi)^2}F^*(s)\int d\cos\theta\ d\varphi T^0(s,t), \qquad \text{(III.1-6)}$$

where $\cos\theta = 1 + \frac{2t}{s}$ is the scattering angle in c.m.f. and φ is the azimuthal angle.

Using the usual definition of the direct and backward partial wave decomposition:

$$T^I(s,t) = 32\pi \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) t_l^I(s), \tag{III.1-7}$$
$$t_l^I(s) = \frac{1}{64\pi} \int_{-1}^{1} T^I(s,t) P_l(\cos\theta)\, d\cos\theta,$$

one obtains the simple relation

$$\mathrm{Disc}F(s) = iF^*(s) t_0^0(s) + \mathcal{O}(\text{multi-particle states}). \tag{III.1-8}$$

Let us suppose that the LLog coefficients of the $\pi\pi$-scattering partial waves are known:

$$t_l^I(s) = \frac{\pi}{2} \sum_{n=1}^{\infty} \hat{S}^n \frac{\omega_{nl}^I}{2l+1} \ln^{n-1}\left(\frac{\mu^2}{-s}\right) + \mathcal{O}(\text{NLLog}). \tag{III.1-9}$$

Substituting this expression together with exp. (III.1-2) (note, that according to our definition, $\mathrm{Disc}\ln\left(\frac{1}{-s}\right) = +i\pi$) into eqn. (III.1-8), and collecting the coefficients of equal powers of F and logarithms, one obtains the relation between LLog coefficients for the scalar FF

$$v_n = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{2} \omega_{n-i,0}^0 v_i. \tag{III.1-10}$$

This equation is an analog of eqn. (II.4-63), but it is obtained without any loop-calculation. Generally speaking, we have obtained the coefficient at the leading $\ln(s)$, while in the previous chapter we have calculated the coefficient at the leading $\ln(\mu^2)$. But since, there is no other dimensional parameters in the matrix element (III.1-1), except s and μ^2, the coefficients at $\ln(s)$ and $\ln(\mu^2)$ are equal. This trick works only in the massless theories, because in the massive theories the term $\ln(\mu^2/m^2)$ could appear. Such terms are "invisible" in the complex s-plane, but contribute to the RG logarithm coefficient. In principle one can obtain the RG LLog behavior of the massive theories with the help of the analytical continuation of the Green function to the complex-mass plane.

The obtained equation (III.1-10) is more simple than eqn. (II.4-63) because eqn. (III.1-10) is written in terms of the physical coefficients. In order to show the equivalence of eqns. (III.1-10) and (II.4-63), one has to express the amplitude LLog coefficients ω_{nl}^I through the "fundamental" RG coefficients ω_{nl}. The amplitude of the $\pi\pi$ scattering has the form (II.3-43), where the amplitudes A, B, and C in the LLog approximation are (II.3-44), (II.3-50) and (II.3-51) respectively. The isospin components of the $\pi\pi$ scattering amplitude in terms of subamplitudes A, B and C

read

$$\begin{aligned} T^0(s,t) &= NA(s,t)+B(s,t)+C(s,t), \\ T^1(s,t) &= B(s,t)-C(s,t), \\ T^2(s,t) &= B(s,t)+C(s,t). \end{aligned} \qquad \text{(III.1-11)}$$

These expressions give the relations between the "fundamental" LLog coefficient ω_{nC} and their isospin projections

$$\begin{aligned} \omega_n^0 &= \omega_n \cdot (NI + U\Omega_n + \Omega_n U), \\ \omega_n^1 &= \omega_n \cdot (U\Omega_n - \Omega_n U), \\ \omega_n^2 &= \omega_n \cdot (U\Omega_n + \Omega_n U). \end{aligned} \qquad \text{(III.1-12)}$$

Substituting ω_n^0 into eqn. (III.1-10) one obtains eqn. (II.4-63). The anomalous dimension, obtained in exp. (II.4-64) by the loop calculation, is the projector on the isospin-0 component of the zero partial wave of the $\pi\pi$-scattering amplitude:

$$Z_S^{n-i,C} = \frac{N}{2}\delta^{C0} + \frac{1+(-1)^C}{2}\Omega_{n-i}^{C0}.$$

The consideration of the vector FF (II.4-66) can be done in the same way. The equation for its LLog coefficients is

$$v_n = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{6}\omega_{n-i,1}^1 v_i. \qquad \text{(III.1-13)}$$

Thus, the anomalous dimension of the vector current is proportional to the isospin-1 projector (III.1-12)

$$Z_V^{n-i,C} = \frac{1+(-1)^C}{6}\Omega_{n-i}^{C1},$$

which is in agreement with the loop calculation of the previous chapter (II.4-68).

Although the obtained equations have passed many tests, we add one more argument to their correctness. The solutions of eqns. (III.1-10) and (III.1-13) can be easily found in terms of generating functions (see appendix C.2 for details) defined as

$$\mathcal{F}_{S(V)}(x) = \sum_{n=0}^{\infty} v_n^{S(V)} x^n, \qquad \text{(III.1-14)}$$

$$W_{S(V)}(x) = \sum_{n=1}^{\infty} \frac{\omega_{n0(1)}^{0(1)}}{2(6)} x^{n-1}. \qquad \text{(III.1-15)}$$

In terms of generating functions, recursive equations (III.1-10) and (III.1-13) turn into the ordinary integral Volterra equations of the second kind with the following solution

$$\mathcal{F}_{S,V}(x) = e^{\int_0^x dy W_{S,V}(y)}. \tag{III.1-16}$$

The FF and the $\pi\pi$-scattering amplitude at the LLog approximation are related to the generation functions as following

$$F_{S,V}(s) = (4\pi)^2 \mathcal{F}_{S,V}\left(\hat{S}\ln\left(\frac{\mu^2}{-s}\right)\right),$$

$$t_{0(1)}^{0(1)}(s) = \pi \hat{S} W_{S,V}\left(\hat{S}\ln\left(\frac{\mu^2}{-s}\right)\right),$$

where we set $2B = 1$ for the scalar FF. Therefore, the FFs at LLog approximation can be expressed through the LLog approximation of $\pi\pi$-scattering partial waves:

$$F_{S,V}(q^2) = \exp\left[\frac{1}{\pi}\int_0^{q^2}\frac{ds}{s} t_{0(1)}^{0(1)}(s)\left(\ln\left(\frac{\mu^2}{-s}\right) - 1\right)\right]. \tag{III.1-17}$$

Making algebraic transformations, one can show that this representation of FF coincides with LLog approximation of the well-known Omnes solution for the dispersion relations of FF [56]

$$F_{S,V}(q^2) = \exp\left[\frac{q^2}{\pi}\int_0^{\infty}\frac{ds}{s}\frac{\delta_{0(1)}^{0(1)}(s)}{s - q^2 - i\varepsilon}\right], \tag{III.1-18}$$

where δ is the phase of the partial wave, i.e. $t_l^I(s) = (2i)^{-1}\left(e^{2i\delta_l^I(s)} - 1\right)$. The Omnes solution is an exact expression for FF in the infrared region. Thus, the recursive equations give the correct infrared behavior of FF.

The similar consideration of FFs can be done in any model. The equations in different models will be similar with (III.1-10,III.1-13), since the unitarity relation and the analytical properties are universal conceptions.

III.2 LLog for $\pi\pi$-scattering

The FF LLog coefficients v_n are given by the recursive equation (III.1-10) through the LLog coefficients ω_{nl}^I of the partial wave t_l^I. However, the coefficients ω_{nl}^I are unknown. Let us extract them from the analytical properties of $\pi\pi$-scattering amplitude.

It is possible to find the corresponding relations without specification of the EFT. One has to fix only some general assumptions for the theory. The particular realization of the EFT influences the initial values of the iteration procedure only.

We constrain the group of the considered theories by the following assumptions:

- The interaction part of the Lagrangian is $\mathcal{O}(\phi^4)$, i.e. the lowest order interaction term has four fields.
- All fields are massless.
- The lowest chiral order of the interaction is $2k$, e.g. $\phi^2\partial^{2k}\phi^2$. Note that $k = 0$ is the case of a pure renormalizable quantum field theory.
- The Lagrangian is invariant under some known global group G of the field transformations.

Also, we assume that the space-time dimension $D = 4$. These constraints define some quite general massless EFT, for example all massless σ-models satisfy these conditions.

Let us consider the S-matric element of the $2 \to 2$ scattering

$$\langle \phi^d(p_4)\phi^c(p_3)|\phi^b(p_2)\phi^a(p_1)\rangle = I + i(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\sum_I P_I^{abcd}T^I(s,t,u), \tag{III.2-19}$$

where I is the identity part of the process.

P_I^{abcd} is the projector of the $2\otimes 2$ product on the I's invariant subspace of the particle representation. According to the Wigner-Eckart theorem it can be represented as the convolution of two Clebsh-Gordon coefficients, see e.g.[57],

$$P_I^{abcd} \simeq \sum_j C^{(Ij)}_{(\lambda a)(\lambda b)} C^{(\lambda c)(\lambda d)}_{(Ij)},$$

where λ is the number of the particle representation, the definition is given up to the normalization constant. For example, the system of 2 pions, which are isospin-1 particles, can be composed of 0-,1- and 2-isospin states, which realize the invariant subspaces. Note, that the $2\otimes 2$ tensor can be in mixed representation, e.g. the proton-pion scattering isospin structure has $Adj \times Fun$ -representation. For the compactness of the notations, we will omit the summation sign over the group indices.

The basis of the projectors is orthogonal and complete

$$P_I^{ab,c'd'}P_{I'}^{d'c',cd} = \delta_{II'}P_I^{ab,cd}, \tag{III.2-20}$$
$$\sum_I P_I^{ab,cd} = \delta^{ad}\delta^{bc}. \tag{III.2-21}$$

The dimension of the invariant subspace d_I can be obtained from the projector using the relation

$$d_I = P_I^{ab,ba}. \tag{III.2-22}$$

The Bose and the Lorentz symmetries imply that the matrix element (III.2-19) is invariant under the simultaneous permutation of particle indices and momenta. That leads to the set of the crossing relations on the amplitudes A^I, namely

$$T^I(s,t,u) = C^{II'}_{st} T^{I'}(t,s,u), \quad \text{(III.2-23)}$$
$$T^I(s,t,u) = C^{II'}_{tu} T^{I'}(s,u,t), \quad \text{(III.2-24)}$$
$$T^I(s,t,u) = C^{II'}_{su} T^{I'}(u,t,s). \quad \text{(III.2-25)}$$

The crossing matrices are defined through the projector P_I as follows

$$C^{II'}_{st} = P^{ab,cd}_I P^{cb,ad}_{I'} \frac{1}{d_I}, \quad \text{(III.2-26)}$$
$$C^{II'}_{tu} = P^{ab,cd}_I P^{ba,cd}_{I'} \frac{1}{d_I}, \quad \text{(III.2-27)}$$
$$C^{II'}_{su} = P^{ab,cd}_I P^{bd,ac}_{I'} \frac{1}{d_I}. \quad \text{(III.2-28)}$$

The crossing matrices satisfy the relations: $C = C^{-1}$ and $C_{st}C_{tu} = C_{tu}C_{su}$.

The structure of singularities for the $2 \to 2$ amplitude is more difficult than the structure of singularities for FF. In the complex s-plane the partial wave amplitude $t^I_l(s)$ has right (s-channel) and left (u-channel) singularities. In the chosen class of theories the only presented singularities are cuts, since 1-particle intermediate state would induce the interaction term with three fields and violate the first assumption on the class of considering theories. In principle, the 1-particle intermediate states can be induced by any odd-number operator, but such terms are subleading in our consideration. If the interaction particles have equal masses m, the s-channel cuts branching points are $s_2 = 4m^2$, $s_3 = 9m^2$ and so on, for the 2-,3-particle, and so on cuts, respectively. The u-channel cuts branching points are $u_k = 4m^2 - s_k$. In the massless limit all branching points flow together at the origin, and the complex s-plane become bisected onto two completely separated semi-planes. In order to avoid this situation we always assume the presence of infinitesimal masses of the particles, which leads to the tiny gap between s and u-channel singularities near the origin. This allows us to operate with the dispersion relations and to make the analytic continuation through this gap from the upper to the downer semiplane.

In the perturbative expansion the sources of the s-(u-)channel cuts are $\ln(s)$ ($\ln(-s)$). Therefore, we have to introduce the logarithms of these two types in the expansion (III.1-9). The total power of the logarithms have to be kept unchanged. The "modified" perturbative expansion of the partial wave can be presented in the following form

$$t^I_l(s) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\hat{S}^n}{2l+1} \sum_{i=0}^{n-1} \alpha^{I,l}_{n,i} \ln^i\left(\frac{\mu^2}{s}\right) \ln^{n-i-1}\left(\frac{\mu^2}{-s}\right) + \mathcal{O}(\text{NLLog}), \quad \text{(III.2-29)}$$

where $\hat{S} = s^k/(4\pi F)^2$. The RG logarithm coefficient ω^I_{nl} is given by

$$\omega^I_{nl} = \sum_{i=0}^{n-1} \alpha^{I,l}_{n,i}. \qquad \text{(III.2-30)}$$

The discontinuity over the s-channel cuts is given by the unitarity relation, which is diagonal in the partial wave and the group subspace indices

$$\mathrm{Disc}\, t^I_l(s) = |t^I_l(s)|^2 + \mathcal{O}(\text{Inelastic part}). \qquad \text{(III.2-31)}$$

The diagonality over the group subspace indices is the consequence of the orthogonality relation (III.2-20). Multi-particle intermediate states do not influence the LLog part, which can be shown in the same way as in (III.1-4). Substituting exp. (III.2-29) and comparing the coefficients at the terms $\hat{S}^n \ln^{n-2}(\mu^2/|s|)$, one obtains the relation

$$\sum_{i=0}^{n-1}(n-i-1)\alpha^{I,l}_{n,i} = \frac{1}{2(2l+1)} \sum_{i=1}^{n-1} \omega^I_{il}\omega^I_{n-i,l}. \qquad \text{(III.2-32)}$$

Relation (III.2-32) does not fix the RG logarithm coefficient, (III.2-30). The additional relation on α can be found from the discontinuity over the u-channel cut. The latter can be found by the analytical continuation of the unitarity relation (III.2-31) to the u-channel area.

A simple dispersion relation for the elastic scattering amplitude can be written in the form

$$\begin{aligned} T^I(s,t) &= \frac{1}{\pi}\int_C ds' \frac{T^I(s',t)}{s'-s} \qquad \text{(III.2-33)} \\ &= \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left(\frac{\delta^{II'}}{s'-s} + \frac{C^{II'}_{su}}{s'-4m^2+t+s}\right) \mathrm{Disc} T^{I'}(s',t), \end{aligned}$$

where for the second equality exp. (III.2-25) was used. To be sure that the dispersion relation is convergent one has to make the subtractions. However, the subtractions do not influence the imaginary part of the amplitude but only on its real part. Thus, we can omit this procedure.

In terms of partial waves (III.1-7) the dispersion relation (III.2-33) has the form

$$\begin{aligned} t^I_l(s) &= \frac{2l+1}{2\pi} \sum_{l'=0}^{\infty} \int_{4m^2-s}^{0} dt \int_{4m^2}^{\infty} ds' \qquad \text{(III.2-34)} \\ & P_l\Big(1+\frac{2t}{s-4m^2}\Big)\Big(\frac{\delta^{II'}}{s'-s} + \frac{C^{II'}_{su}}{s'-4m^2-s-t}\Big) P_{l'}\Big(1+\frac{2t}{s'-4m^2}\Big)\mathrm{Disc}\, t^{I'}_{l'}(s'). \end{aligned}$$

On the right-hand side of dispersion relation (III.2-34) the discontinuity is taken over the s-channel cuts. Therefore, the unitarity relation (III.2-31) can be used. Eqn. (III.2-34) is an analog of the famous Roy equation for arbitrary symmetry group. The original Roy equation was derived for the $\pi\pi$-scattering amplitude in [58], or for the $SU(2)$-symmetric models.

Taking the discontinuity of exp. (III.2-34) in the $s<0$ region one obtains

$$\begin{aligned}\mathrm{Disc}\, t_l^I(s) &= \sum_{l'=0}^{\infty} C_{su}^{II'} \frac{2(2l'+1)}{s-4m^2} \int_{4m^2}^{4m^2-s} ds' \\ & P_l\Big(\frac{s+2s'-4m^2}{4m^2-s}\Big) P_{l'}\Big(\frac{2s+s'-4m^2}{4m_\pi^2-s'}\Big) \Big(|t_{l'}^{I'}(s')|^2 + \mathcal{O}(\text{Inelastic part})\Big) .\end{aligned} \tag{III.2-35}$$

This is the analytic continuation of the unitarity relation (III.2-31) onto the u-channel cut. Now the limit $m^2 \to 0$ can be taken.

Substituting ansatz (III.2-29) into eqn. (III.2-35) and collecting the coefficients at the appropriate terms one obtains

$$\sum_{i=0}^{n-1} i\alpha_{n,i}^{I,l} = \sum_{l'=0}^{kn} \frac{C_{su}^{II'}}{2l'+1} \sum_{i=1}^{n-1} \omega_{il'}^{I'} \omega_{n-i,l'}^{I'} (-1)^{l+l'} \Omega_{kn}^{l'l}, \tag{III.2-36}$$

where the matrix Ω was defined by exp. (II.3-48).

In eqn. (III.2-36) the matrix Ω appears as the LLog part of the integral

$$\int_0^{-s} ds' \frac{s'^n}{s} \ln^{n-2}(\frac{\mu^2}{s'}) P_l\Big(\frac{s+2s'}{-s}\Big) P_{l'}\Big(\frac{2s+s'}{-s'}\Big) =$$

$$\frac{-1}{2l+1} s^n \ln^{n-2}\left(\frac{\mu^2}{s}\right) (-1)^{l+l'} \Omega_n^{l'l} + \mathcal{O}\left(s^n \ln^{n-3}\left(\frac{\mu^2}{s}\right)\right).$$

Therefore, the matrix Ω_n can be interpreted as the crossing matrix for the n-th order of the chiral expansion of the partial wave t_l^I in the LLog approximation. The matrix Ω performs the $(s \leftrightarrow t)$ crossing transformation. The $(t \leftrightarrow u)$ and $(s \leftrightarrow u)$ crossing transformations are performed by matrices U and $U\Omega U$ respectively, where the matrix $U = \mathrm{diag}\{1,-1,1,...\}$. In such terms, exp. (III.2-35) has a simple meaning: it is the crossing transformation of relation (III.2-31) simultaneously in the group and momenta spaces.

The coefficients of the RG logarithm (III.2-30) are given by the sum of exps. (III.2-32) and (III.2-36)

$$\omega_{nl}^I = \Big(\frac{1+(-1)^l C_{tu}}{2}\Big)^{IJ} \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{l'=0}^{kn} \Big(\frac{\delta^{ll'}}{2} + C_{su}(-1)^{l+l'} \Omega_{kn}^{l'l}\Big)^{JI'} \frac{\omega_{il'}^{I'} \omega_{n-i,l'}^{I'}}{2l'+1}. \tag{III.2-37}$$

The factor $\frac{1}{2}(1+UC_{tu})$ makes the symmetrization over the $(t \leftrightarrow u)$ parity. This is the recursive equation for the LLog coefficient of the $2 \to 2$ scattering amplitude.

Eqn. (III.2-37) is the complete analog of eqn. (II.3-36), which is the consequence of the renorminvariance. Thus, in some sense, the RG evolution and RGEs are the consequence of the amplitude analytic properties. The β-function of the theory is just a combination of the crossing matrices. Using the fact that the first term in the kernel of eqn (III.2-37) is invariant under the $(t \leftrightarrow u)$ crossing symmetry, one can rewrite eqn. (III.2-37) in the following form

$$\vec{\omega}_n = \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{2}\left(\mathfrak{I} + \mathfrak{C}_{st} + \mathfrak{C}_{su}\right)\vec{\omega}_i\vec{\omega}_{n-i}, \qquad \text{(III.2-38)}$$

where $\mathfrak{C}$ are the full crossing matrices, i.e. they perform the crossing transformation in the momenta and group spaces simultaneously, $\vec{\omega}_n$ is the vector in both spaces, and we have subtracted the factor $(2l+1)^{-1}$ somewhere. In this form the analogy with the one-loop diagrams in fig.II.3. is obvious: the factor one-half is the symmetry factor of the diagrams and the crossing transformations transform the diagrams as a whole.

In case of the renormalizable theory when $k = 0$, one has the simple expression for the β-function of the coupling,

$$\beta = \frac{1}{2}(I + C_{st} + C_{su}), \qquad \text{(III.2-39)}$$

where we have used that $\Omega_0 = 1$. It is not the usual form of β-function defined in QFT because the Lagrangian is not formulated in terms of the invariant subspaces, but in terms of the crossing symmetrical linearly independent subspaces. The transition to the Lagrangian degrees of freedom shows that our system of coefficients ω^I_{nl} and β-functions is overfulled. The Lagrangian is crossing-symmetrical by itself without any addition specification. Therefore, eqn. (III.2-37) contains at least three times more variables.

Eqn. (III.2-37) is diagonal in all auxiliary indices. That essentially decreases the computer working time during the numerical calculations, in spite of the increasing of the number of variables. It also gives some hope to solve eqn. (III.2-37) analytically. At this point we want to add some circumstances which may play a key role for a possible solution. The kernel of eqn. (III.2-38) has projector properties in an appropriate space, i.e. $\beta \times \beta = \frac{3}{2}\beta$, which can be easily seen in the renormalizable version of kernel (III.2-39). Moreover, since the β-functions is a projector, the coefficients ω are invariant under the multiplication on any crossing matrix in both isospin and partial wave spaces. This invariance fixes most parts of the values of the different partial and iso-spin components ω^I_{nl}'s at the same order n. However, it does not fix the dependence completely, and the number of independent ω's still grows with n. Such consideration implies the equations with many symmetries, which are (in our opinion) easier solvable.

Models with mixed renormalizable and non-renormalizable interactions Let us consider the theory with mixed renormalizable and non-renormalizable interactions. The Lagrangian of such a theory has two interaction terms, with $k = 0$ (renormalizable term) and with $k > 0$ (non-renormalizable term). Let $\lambda(\frac{1}{F^2})$ be a coupling of the (non-)renormalizable part of the initial Lagrangian. The renormalizable interaction can only have the following form $T^{abcd}\phi^a\phi^b\phi^c\phi^d$, where T^{abcd} is a group tensor.

The perturbative expansion in such a mixed theory is a double expansion over the dimensional parameter $\hat{S}$ and the dimensionless coupling λ. Using the partial wave parametrization one can write the perturbative expansion in the form

$$\begin{aligned} T^I(s,t,u) &= (4\pi)^2 \sum_{n,m=0}^{\infty} \sum_{l=0}^{kn} \hat{S}^n \hat{\lambda}^m \times \\ & \sum_{i=0}^{n+m-1} \alpha^{I,l}_{m,n,i} \ln^i \left(\frac{\mu^2}{s}\right) \ln^{n+m-1-i} \left(\frac{\mu^2}{-s}\right) P_l\left(1+\frac{2t}{s}\right) + \mathcal{O}(\text{NLLog}), \end{aligned} \tag{III.2-40}$$

where $\hat{\lambda} = \frac{\lambda}{(4\pi)^2}$, and $\alpha^{I,l}_{00l} \equiv 0$. The analytical properties of the amplitude do not depend on the "intrinsic" structure of the Lagrangian. Hence, repeating the discourse of the previous section for the series (III.2-40) one obtains that the LLog coefficients ω^I_{nml} satisfy the equation

$$\omega^I_{nml} = \frac{\left(\frac{1+(-1)^l C_{tu}}{2}\right)^{IJ}}{n+m-1} \sum_{i,j=0}^{n,m} \sum_{l'=0}^{kn} \left(\frac{\delta^{ll'}}{2} + C_{su}(-1)^{l+l'}\Omega^{l'l}_{kn}\right)^{JI'} \frac{\omega^{I'}_{ijl'}\omega^{I'}_{n-i,m-j,l'}}{2l'+1}, \tag{III.2-41}$$

where $\omega_{00l} \equiv 0$.

Pure renormalizable and pure non-renormalizable parts are split up in the equation, i.e. the coefficients ω^I_{n0l} and ω^I_{0m0} are not mixed in eqn. (III.2-41) with other coefficients. Indeed, one can not obtain the dimensionless result from the diagram with any dimensional non-renormalizable vertex, and visa versa, the highest dimensional contribution can not be fulfilled in the presence of the dimensionless λ. One can see that in the cases $m(n) = 0$, eqn. (III.2-41) is the usual "one-parameter-expansion" equation (III.2-37) with $k = 0(k)$.

The structure of eqn. (III.2-41) reflects the renormalization structure of the theory. In a such mixed theory, although it is still non-renormalizable, the β-functions contain infinite number of terms. We have not met such structure yet, since the dimensional object can be constructed from other dimensional objects in a finite number of manners, e.g. (II.2-12). In the presence of the additional renormalizable vertex, one can couple an infinite "tail" of the renormalizable graphs to every graph.

The trace of the partial-renormalizabillity of the theory is the m-independence of eqn. (III.2-41) kernel. Using this property the renormalizable part can be resumed at every order of the "non-renormalizable" expansion over $\hat{S}$. Let us demonstrate the resummation technique on the equation of (III.2-41)-type, but without auxiliary indices:

$$\omega_{n,m} = \frac{1}{n+m-1} \sum_{i,j=0}^{n,m} \beta_n \omega_{i,j} \omega_{n-i,m-j}. \tag{III.2-42}$$

Introducing the generation function $\omega_n(x) = \sum_{m=0}^{\infty} x^{n+m-1} \omega_{n,m}$ for the renormalizable part, one obtains the integral equation (the similar equations are considered in appendix C.2 in details)

$$\omega_n(x) - \omega_n(0) = \int_0^x \sum_{i=1}^{n-1} \beta_n \omega_i(y) \omega_{n-i}(y) \; dy.$$

The boundary conditions for the equation are $\omega_0(0) = \omega_{0,1}$, $\omega_1(0) = \omega_{1,0}$ and $\omega_{n>1}(0) = 0$. The first two iterations give

$$\omega_0(x) = \frac{\omega_{0,1}}{1-\beta_0 x}, \quad \omega_1(x) = \frac{\omega_{10}}{(1-\beta_0 x)^{\beta_1/\beta_0}}.$$

The function ω_0, which describes the pure renormalizable part of the interaction, reproduces the usual renormalizable running of the 4-point amplitude at the LLog approximation. The higher orders of the resumed non-renormalizable expansion can be found recursively,

$$\omega_n(x) = \int_0^x \sum_{i=1}^{n-1} \omega_i(y) \omega_{n-i}(y) \left(\frac{1-\beta_0 \omega_{0,1} y}{1-\beta_0 \omega_{0,1} x} \right)^{\beta_n/\beta_0}.$$

The auxiliary indices makes the equation more complicated, but the summation scheme remains unchanged.

The theories in $D > 4$ dimensions For the consideration of a theory in the $D > 4$ space-time dimensions we have to generalize the partial wave basis. The complete and orthogonal basis in the $(D-1)$-dimension space is the basis of the Gegenbauer polynomials, [80]. We define the partial wave expansion in an arbitrary even dimension D as

$$\begin{aligned} T^I(s,t) &= 64\pi \sum_{l=0}^{\infty} \frac{2l+D-3}{2} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} C_l^{\frac{D-3}{2}}(\cos\theta) t_l^I(s), \qquad \text{(III.2-43)} \\ t_l^I(s) &= \frac{1}{64\pi} \int_0^{\pi} \sin^{D-3}\theta \frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} \frac{2^{D-4} l!}{\Gamma(l+D-3)} T^I(s,\cos\theta) C_l^{\frac{D-3}{2}}(\cos\theta) d\theta, \end{aligned}$$

where $C_l^{\frac{D-3}{2}}(z)$ is a Gegenbauer polynom. At $D=4$ these relations are equal to the usual partial wave decomposition (III.1-7). The unitarity relation for the D-dimensional partial waves has the form

$$\mathrm{Im} t_l^I(s) = \pi^{\frac{D-4}{2}} \left(\frac{s}{(4\pi)^2}\right)^{\frac{D-4}{2}} |t_l^I(s)|^2 + \mathcal{O}(\text{Inelastic part}),\ s>0. \qquad \text{(III.2-44)}$$

The detailed derivation of the unitarity relation (III.2-44) can be found in Appendix D. The crossing relations (III.2-23)-(III.2-25) do not depend on the space-time dimension, therefore, the evaluation of the discontinuity over the u-channel cut is straightforward. One obtains ($s<0$)

$$\begin{aligned}\mathrm{Im}\, t_l^I(s) = \sum_{l'=0}^{\infty} C_{su}^{II'} \frac{2^{D-3}(2l'+D-3)}{\Gamma(l+D-3)} \frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi} l! \int_{4m^2}^{4m^2-s} \frac{ds'}{s-4m^2} \times \qquad \text{(III.2-45)}\\ \left[\frac{4s'(4m^2-s-s')}{(s-4m^2)^2}\right]^{\frac{D-4}{2}} C_l^{\frac{D-3}{2}}\left(\frac{s+2s'-4m^2}{4m^2-s}\right) C_{l'}^{\frac{D-3}{2}}\left(\frac{2s+s'-4m^2}{4m^2-s'}\right) \mathrm{Im}\, t_{l'}^{I'}(s').\end{aligned}$$

At an arbitrary space-time dimension D a diagram, which contributes to the perturbative expansion, has the momenta dimension dictated by exp. (I.1-6). Therefore, the perturbative expansion of the scattering amplitude can be written in the form

$$\begin{aligned}T^I(s,t) &= \frac{2^{D-4}(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} \cdot \qquad \text{(III.2-46)}\\ &\sum_{n=1}^{\infty} \sum_{l=0}^{nk+\frac{D-4}{2}(n-1)} \hat{S}^n s^{\frac{d-4}{2}(n-1)} \omega_{nl}^I \ln^{n-1}\left(\frac{\mu^2}{|s|}\right) C_l^{\frac{D-3}{2}}\left(1+\frac{2t}{s}\right) + \mathcal{O}(\text{NLLog}),\end{aligned}$$

$$\hat{S} = \frac{\sqrt{\pi} s^k}{(4\pi)^{\frac{D}{2}} 2^{D-4} \Gamma\left(\frac{D-3}{2}\right) F^2}.$$

Substituting T^I into eqns. (III.2-44) and (III.2-45) one obtains that the LLog coefficients ω satisfy the relation

$$\begin{aligned}\omega_{nl}^I &= \left(\frac{1+(-1)^l C_{tu}}{2}\right)^{IJ} \cdot \qquad \text{(III.2-47)}\\ &\frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{l'=0}^{nk+\frac{D-4}{2}(n-1)} \left(\frac{\delta^{ll'}}{2} + C_{su}(-1)^{l+l'} \Omega^{l'l}_{nk+\frac{D-4}{2}(n-1),D}\right)^{JI'} \frac{\omega_{il'}^{I'}\omega_{n-i,l'}^{I'}}{2l'+D-3},\end{aligned}$$

where $\Omega_{n,D}$ is the straightforward generalization of the crossing matrix Ω_n to the arbitrary D dimension (its properties and explicit form can be founded in Appendix

B.2)

$$\Omega^{AB}_{n,D} = \frac{2B+D-3}{2}\frac{2^{D-4}B!}{\Gamma(B+D-3)}\frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi} \int_{-1}^{1}(1-z^2)^{\frac{D-4}{2}}\left(\frac{z-1}{2}\right)^n C_A^{\frac{D-3}{2}}\left(\frac{z+3}{z-1}\right)C_B^{\frac{D-3}{2}}(z)dz. \tag{III.2-48}$$

Note, that the volume factor $\frac{\Gamma((D-3)/2)}{\sqrt{\pi}}$ in definition (III.2-46) is already subtracted from the β-function in eqn. (III.2-47).

The dimensional regularization gives the final result only in the area of one particular value of D although it operates with a formally arbitrary D. Eqn. (III.2-47) gives us the possibility to investigate the perturbative expansion at the truly arbitrary D.

Relation between ω's The important technical question is how to compare the results obtained in the "physical" terms ω^I_{nl} with the "fundamental" ones, which are used in the loop calculation (II.3-44).

The result of the perturbative calculation is proportional to some linear combination of the invariant subspaces:

$$\langle\phi^c(p_4)\phi^d(p_3)|S-I|\phi^b(p_2)\phi^a(p_1)\rangle \sim \sum_j \tilde{R}^{abdc}_j A_j(s,t,u), \tag{III.2-49}$$

where $\tilde{R}$ denotes the structures of the Lagrangian. For example, the Lagrangian of $O(N)$ model (II.3-43) contains the structure $\delta^{ab}\delta^{cd}$ and its crossing combinations. Another example is $SU(N)$-symmetric Lagrangian (II.5-80), it has six independent combinations of Kronecker deltas and structure constants d^{abc}.

Exp. (III.2-49) has to be in the crossing symmetric form, since it is the S-matrix element. Therefore, we can rewrite it as

$$\langle\phi^c(p_4)\phi^d(p_3)|S-I|\phi^b(p_2)\phi^a(p_1)\rangle \sim \sum_j \big(R^{abdc}_j A_j(s,t,u) + R^{abcd}_j A_j(s,u,t) + R^{adbc}_j A_j(u,t,s) + 21\ \text{combinations}\big)\,. \tag{III.2-50}$$

There are 24 possible index combinations and eight momenta combinations, because the simultaneous interchange of the two fields in the same channel changes nothing in the momenta structure, but gives another index structure. Often the coefficients R are symmetric over this interchange, but it is not a general case. For example, the amplitude in a matrix model (II.5-83) is not $(u \leftrightarrow t)$-symmetrical.

Under the crossing transformation, the coefficients ω transform with the help of the crossing matrices Ω and U, see the details in the text after eqn. (III.2-36). Using the properties of matrices Ω (II.3-49) and the orthogonality of P_I-basis one finds the relations between the LLog coefficients of amplitudes (III.2-50)($\omega_n(j)$) and the LLog coefficients of amplitudes of invariant subspace (ω_n^I) (III.2-19):

$$\omega_n^I = \frac{1}{d_I}\sum_j \omega_n(j)\cdot\mathcal{P}(j)_n^I, \tag{III.2-51}$$

where j enumerates independent charges of a theory. The coefficients $P(j)_n^I$ are given by the following expression

$$\mathcal{P}(j)_n^I = \Big(A_j^I I + \tilde{A}_j^I U + B_j^I U\cdot\Omega_n + \tilde{B}_j^I\Omega_n + C_j^I\Omega_n\cdot U + \tilde{C}_j^I U\cdot\Omega_n\cdot U\Big), \tag{III.2-52}$$

where

$$\begin{aligned}
A_j^I &= P_I^{dc,ba}\big(R_j^{abcd}+R_j^{badc}+R_j^{dcba}+R_j^{cdab}\big),\\
\tilde{A}_j^I &= P_I^{dc,ba}\big(R_j^{abdc}+R_j^{bacd}+R_j^{dcab}+R_j^{cdba}\big),\\
B_j^I &= P_I^{dc,ba}\big(R_j^{adbc}+R_j^{bcad}+R_j^{dacb}+R_j^{cbda}\big),\\
\tilde{B}_j^I &= P_I^{dc,ba}\big(R_j^{adcb}+R_j^{bcda}+R_j^{dabc}+R_j^{cbad}\big),\\
C_j^I &= P_I^{dc,ba}\big(R_j^{acdb}+R_j^{bdca}+R_j^{dbac}+R_j^{cabd}\big),\\
\tilde{C}_j^I &= P_I^{dc,ba}\big(R_j^{acbd}+R_j^{bdac}+R_j^{dbca}+R_j^{cadb}\big).
\end{aligned}$$

III.3 $O(N)$-type models

The fascination of the presented method is that for the investigation of different models with the same type of symmetry one needs to calculate only one algebraic object, namely, the crossing matrices. As an example, let us consider the simplest symmetry configuration. Let the fields ϕ be in the fundamental representation of $O(N)$-group. For such a class of models we have several checking points, namely:

- The renormalizable model $O(N)$-symmetric ϕ^4-model is given by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi^a\partial^2\phi^a - \frac{\lambda}{8}\phi^2\phi^2, \tag{III.3-53}$$

  and its LLog behavior is well-known, e.g. [59],[81].

- The $O(N+1)/O(N)$ σ-model (II.2-5) at the finite order of the Lagrangian expansion also can be treated as the $O(N)$-symmetrical model

$$\mathcal{L}_2 = -\frac{1}{2}\phi^a\partial^2\phi^a - \frac{1}{8F^2}\phi^2\partial^2\phi^2 + \mathcal{O}(\pi^6). \tag{III.3-54}$$

  This model was considered in the previous chapter by the RG method. There are many other independent points of its checking, see the list of them after exp. (II.2-6).

- The large-N behavior of $O(N)$-symmetric models is well-investigated. Diagrammatically the large-N limit is given by the chain diagram, see fig.2.5.

There are three invariant subspaces of the $Fund. \times Fund.$ tensor on the $O(N)$-group. We enumerate these subspaces by the index $I = 0, 1, 2$. Such naming corresponds to the value of the isospin in s-channel for the case $N = 3$. The projectors on invariant subspaces are

$$\begin{aligned} P_0^{ab,cd} &= \frac{1}{N}\delta^{ab}\delta^{cd}, \\ P_1^{ab,cd} &= \frac{1}{2}(\delta^{ad}\delta^{bc} - \delta^{ad}\delta^{bc}), \\ P_2^{ab,cd} &= \frac{1}{2}(\delta^{ad}\delta^{bc} + \delta^{ad}\delta^{bc}) - \frac{1}{N}\delta^{ab}\delta^{cd}. \end{aligned} \tag{III.3-55}$$

The dimensions of these subspaces are

$$d_{\{0,1,2\}} = \left\{1, \frac{N(N-1)}{2}, \frac{(N+2)(N-1)}{2}\right\}.$$

After simple algebraic operations one obtains the crossing matrices (III.2-26)-(III.2-28)

$$C_{tu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; C_{st} = \begin{pmatrix} \frac{1}{N} & \frac{N-1}{2} & \frac{\tilde{N}}{2N} \\ \frac{1}{N} & \frac{1}{2} & -\frac{N+2}{2N} \\ \frac{1}{N} & -\frac{1}{2} & \frac{N-2}{2N} \end{pmatrix}, \; C_{su} = \begin{pmatrix} \frac{1}{N} & -\frac{N-1}{2} & \frac{\tilde{N}}{2N} \\ -\frac{1}{N} & \frac{1}{2} & \frac{N+2}{2N} \\ \frac{1}{N} & \frac{1}{2} & \frac{N-2}{2N} \end{pmatrix}, \tag{III.3-56}$$

where $\tilde{N} \; = \; N^2 + N - 2$.

Substituting matrices (III.3-56) into eqn. (III.2-47) one obtains the LLogs coefficients for the required theory. The relation between the "physical" and "fundamental" LLog coefficients is given by (III.2-51). There is only one combination of indices in the $O(N)$ symmetric Lagrangians (see e.g. (III.3-53))

$$R^{ab,dc} = \delta^{ab}\delta^{cd}.$$

Using relation (III.2-52) one finds the list of projectors of the amplitudes A on the amplitudes T^I, namely (compare with (III.1-11) and (III.1-12))

$$\begin{aligned} \mathcal{P}_n^0 &= \big(I + U\big) \cdot \big(NI + U \cdot \Omega + \Omega \cdot U\big), \\ \mathcal{P}_n^1 &= \big(I + U\big) \cdot \big(U \cdot \Omega - \Omega \cdot U\big), \\ \mathcal{P}_n^2 &= \big(I + U\big) \cdot \big(U \cdot \Omega + \Omega \cdot U\big). \end{aligned} \tag{III.3-57}$$

From this expression it follows that the "fundamental" LLog coefficient that gives the LLog evolution of the coupling constant, relates to the coefficients, which give the LLog behavior of the amplitudes, as follows:

$$\omega_{nl} = \frac{\omega^0_{nl} - \omega^2_{nl}}{N}. \tag{III.3-58}$$

The analytic properties of the amplitude fix only the relations between the logarithm coefficients (III.2-37), which have to be satisfied unless the unitarity would be broken. The particular values of the coefficients are dictated by the initial values of the iteration. The initial values on their own turn are consequences of the tree order of the amplitude. The Lagrangian can be always redefined in such a way that $\omega_{10} = 1$ (and $\omega_{1,l\neq 0} = 0$). That satisfies the definition of the coupling constant with the coefficients $\frac{1}{8}$, e.g. (II.3-37),(III.3-53). The boundary conditions for eqn. (III.2-37) depend on the parameters k and D of the theory. For example, for $k = 1$ from (III.3-57) one obtains that

$$\omega^0_{10} = N - 1, \quad \omega^0_{11} = \frac{1}{D-3}, \quad \omega^2_{10} = -1, \tag{III.3-59}$$

and all other $\omega^I_{1l} = 0$.

With the help of relation (III.3-58) one can rewrite eqn. (III.2-37) in form of eqn. (II.3-36). The result completely agrees with the calculations of β-function (II.3-52) in the previous chapter.

Let us consider the renormalizable ϕ^4 model in $D = 4$, (III.3-53). In this case the main index of matrix Ω is zero. Using $\Omega^{l'l}_0 = \delta^{l0}\delta^{l'0}$, one finds that only zero-partial waves appear in eqn. (III.2-37). The β-function is a number

$$\beta = \frac{N+8}{2}.$$

This result is well-known, e.g. [59]. The β-function does not depend on n, as it should be in the renormalizable theory. Thus, the solution of eqn. (II.3-36) is $\omega_n = \beta^{n-1}\omega^n_1$. That leads to the LLog behavior of the 4-point amplitude

$$A(s,t) = \sum_{n=1}^{\infty} \omega_n \lambda^n \ln^{n-1}\left(\frac{\mu^2}{|s|}\right) = \frac{\lambda}{1 + \frac{N+8}{2(4\pi)^2}\lambda \ln\left(\frac{\mu^2}{|s|}\right)},$$

which coincides with the LLog running of the coupling λ.

At $N = 1$ the $O(N)$ group space contains only one element, which corresponds to the zeroth invariant subspace. At $k = 1$ the boundary condition for the iterations is $\omega^0_1 = 0$, (III.3-59). Therefore, all LLog coefficients are zero and the tree order is also zero. Indeed, at $k = 1$ the $O(N)$ σ-model is a free theory in any space-time dimension.

It is also interesting to consider the case of the ϕ^4-model in D dimensions, i.e. we put $N = 1$ and $k = 0$. The recursive equation can be rewritten in the form

$$\omega_{nl} = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{l'=0}^{\frac{D-4}{2}(n-1)} \frac{\omega_{i,l'}\omega_{n-i,l'}}{2(2J+D-3)}\left(I + \Omega_{\frac{d-4}{2}(n-1)} + U\Omega_{\frac{d-4}{2}(n-1)}U\right)^{l'l} \quad \text{(III.3-60)}$$

with the initial value $\omega_{10} = 3$.

This equation has simple symmetry properties. Its solution is invariant under the multiplication on Ω or $U\Omega U$. This is the trace of the complete crossing symmetry of the amplitude. The amplitude $A(s,t,u)$ is a symmetric function in all its variables. This invariance is explicitly presented in the momentum expansion of the amplitude, i.e. in the Tailor expansion over s and t. We define the LLog coefficient χ_{nk} as a coefficient instead of $t^k s^{\frac{(D-4)}{2}(n-1)-k}$ structure in the amplitude. The transition from ω to χ and back is given by

$$\begin{aligned} \chi_{nk} &= \sum_{l=k}^{\frac{D-4}{2}(n-1)} \frac{4^l\Gamma\left(\frac{D-3}{2}+k\right)\Gamma\left(l+k+D-3\right)}{k!\Gamma\left(\frac{D-3}{2}\right)\Gamma(2k+D-3)(l-k)!}\omega_{nl} \\ \omega_{nl} &= \sum_{k=0}^{\frac{D-4}{2}(n-1)} \Omega^{0,l}_{k,D}\chi_{nk}. \end{aligned} \quad \text{(III.3-61)}$$

Thus, the symmetry of eqn. (III.3-60) leads to the two constrains on χ:

$$\chi_{n,k} = \chi_{n,n-k}, \quad \chi_{nk} = \sum_{k=\alpha}^{n}(-1)^k\binom{k}{\alpha}\chi_{n\alpha}.$$

In particular, from this expression follows that at $D = 6$ the one-loop correction is a totally symmetric function of (s,t,u). Therefore, it is zero, since $(s+t+u) = 0$. There are no LLog corrections to the 4-point amplitude at $D = 6$ and, indeed, the equation for ω gives zeros at $D = 6$ at all orders.

In appendix E we present the tables of the values of the LLog coefficients ω^I_{nl} if the $O(N+1)/O(N)$ σ-model and ϕ^4-model are in $D = 6$.

III.4 $SU(N)$-type models

For the completeness let us give the expressions for the needed ingredients of the recursive equation in the $SU(N)$-symmetric models. Let us consider the fields in the adjoint representation of $SU(N)$ group, e.g. the Lagrangian of massless ChPT (II.5-79). The $Adj \times Adj$ space has seven invariant subspaces. These are (anti)symmetric subspaces, trace(less) subspaces, and subspaces with a mixture of the symmetries (the full procedure of decomposition can be found in [57]). The projectors can be built as following

$$P_1^{ab,cd} = \frac{1}{N^2-1}\delta_{ab}\delta_{cd}, \tag{III.4-62}$$

$$P_2^{ab,cd} = \frac{2N}{N^2-4}\Big(\langle abcd\rangle + \langle abdc\rangle + \langle bacd\rangle + \langle badc\rangle - \frac{1}{N}\delta^{ab}\delta^{cd}\Big),$$

$$\begin{aligned}
P_3^{ab,cd} &= \frac{1}{4}\Big(\delta^{ad}\delta^{bc} + \delta^{ac}\delta^{bd}\Big) \\
&\quad -\frac{1}{2N(N-1)}\delta^{ab}\delta^{cd} - \frac{1}{2}\Big(\langle acbd\rangle + \langle bcad\rangle + \langle adbc\rangle + \langle bdac\rangle\Big) \\
&\quad -\frac{1}{N-2}\Big(\langle abcd\rangle + \langle abdc\rangle + \langle bacd\rangle + \langle badc\rangle - \frac{1}{N}\delta^{ab}\delta^{cd}\Big), \\
P_4^{ab,cd} &= \frac{1}{4}\Big(\delta^{ad}\delta^{bc} + \delta^{ac}\delta^{bd}\Big) - \frac{1}{2N(N+1)}\delta^{ab}\delta^{cd} \\
&\quad +\frac{1}{2}\Big(\langle acbd\rangle + \langle bcad\rangle + \langle adbc\rangle + \langle bdac\rangle\Big) \\
&\quad -\frac{1}{N+2}\Big(\langle abcd\rangle + \langle abdc\rangle + \langle bacd\rangle + \langle badc\rangle - \frac{1}{N}\delta^{ab}\delta^{cd}\Big), \\
P_5^{ab,cd} &= \frac{2}{N}\Big(\langle abcd\rangle - \langle abdc\rangle - \langle bacd\rangle + \langle badc\rangle\Big), \\
P_6^{ab,cd} &= \frac{1}{4}\Big(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\Big) + \frac{1}{2}\Big(\langle acbd\rangle - \langle bcad\rangle - \langle adbc\rangle + \langle bdac\rangle\Big) \\
&\quad -\frac{1}{N}\Big(\langle abcd\rangle - \langle abdc\rangle - \langle bacd\rangle + \langle badc\rangle\Big), \\
P_7^{ab,cd} &= \frac{1}{4}\Big(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\Big) - \frac{1}{2}\Big(\langle acbd\rangle - \langle bcad\rangle - \langle adbc\rangle + \langle bdac\rangle\Big) \\
&\quad -\frac{1}{N}\Big(\langle abcd\rangle - \langle abdc\rangle - \langle bacd\rangle + \langle badc\rangle\Big),
\end{aligned}$$

where $\langle abcd\rangle = \mathrm{tr}(t^a t^b t^c t^d)$, t^a is the generator of the $SU(N)$ group. The dimension of the invariant subspaces can be founded with the help of exp. (III.2-22)

$$d_I = \Big\{1, N^2-1, \frac{N^2(N+1)(N-3)}{4}, \frac{N^2(N-1)(N+3)}{4},$$
$$N^2-1, \frac{(N^2-4)(N^2-1)}{4}, \frac{(N^2-4)(N^2-1)}{4}\Big\}.$$

Therefore, the crossing matrices are

$$C_{su} = \begin{pmatrix} \frac{1}{N^2-1} & 1 & \frac{(N-3)N^2}{4(N-1)} & \frac{N^2(N+3)}{4(N+1)} & -1 & 1-\frac{N^2}{4} & 1-\frac{N^2}{4} \\ \frac{1}{N^2-1} & \frac{N^2-12}{2(N^2-4)} & -\frac{(N-3)N^2}{4(N^2-3N+2)} & \frac{N^2(N+3)}{4(N^2+3N+2)} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{N^2-1} & \frac{1}{2-N} & \frac{N^2-N+2}{4N^2-12N+8} & \frac{N+3}{4N+4} & -\frac{1}{N} & \frac{N+2}{4N} & \frac{N+2}{4N} \\ \frac{1}{N^2-1} & \frac{1}{N+2} & \frac{N-3}{4(N-1)} & \frac{N^2+N+2}{4N^2+12N+8} & \frac{1}{N} & \frac{N-2}{4N} & \frac{N-2}{4N} \\ \frac{1}{1-N^2} & -\frac{1}{2} & -\frac{(N-3)N}{4(N-1)} & \frac{N(N+3)}{4(N+1)} & \frac{1}{2} & 0 & 0 \\ \frac{1}{1-N^2} & \frac{2}{N^2-4} & \frac{(N-3)N}{4(N^2-3N+2)} & \frac{N(N+3)}{4(N^2+3N+2)} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{1-N^2} & \frac{2}{N^2-4} & \frac{(N-3)N}{4(N^2-3N+2)} & \frac{N(N+3)}{4(N^2+3N+2)} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \tag{III.4-63}$$

$$C_{tu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{III.4-64}$$

There are two independent structures invariant under the $SU(N)$-transformation, namely $T_1^{abcd} = \langle abcd\rangle$ and $T_2^{abcd} = \langle ab\rangle\langle cd\rangle$. Their LLog coefficients are given by ω_n and v_n for T_1 and T_2 respectively. The coefficients ω_n and v_n satisfy eqn. (II.5-81).

The correspondence between $\omega_n(v_n)$ and ω_n^I is given by exp. (III.2-51) with

$$A_1 = \tilde{A}_1 = \begin{pmatrix} N - \frac{1}{N} \\ \frac{N}{2} - \frac{2}{N} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; B_1 = \tilde{B}_1 = \begin{pmatrix} \frac{N}{2} - \frac{1}{N} \\ \frac{n}{4} - \frac{2}{N} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{N}{4} \\ 0 \\ 0 \end{pmatrix}, \quad C_1 = \tilde{C}_1 = \begin{pmatrix} \frac{N}{2} - \frac{1}{N} \\ \frac{N}{4} - \frac{2}{N} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{N}{4} \\ 0 \\ 0 \end{pmatrix},$$

$$A_2 = \tilde{A}_2 = \begin{pmatrix} N^2 - 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; B_2 = \tilde{B}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; C_2 = \tilde{C}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

From this correspondence, one obtains the inverse relation between "fundamental" and "physical" LLog coefficients

$$\omega_n = \frac{1}{4}\Big(\frac{4}{N}\big(\omega_n^5 - \omega_n^6\big) + \omega_n^4 - \omega_n^3\Big)\cdot\Omega_n,\ \ v_n = \frac{1}{8}\Big(\frac{\omega_n^3+\omega_n^4}{2} + \omega_n^6\Big)\cdot\Omega_n.$$

A direct calculation shows that the numbers obtained by eqns. (II.5-81)) and (III.2-37) coincide. We stress that although crossing matrices (III.4-63)-(III.4-64) are large, the direct loop-calculation of β-functions is more difficult.

Let us consider the large-N limit of the $SU(N)$-symmetric theories. In the previous chapter we have already considered this limit and we found that the coefficient v_n is subleading, and the large-N behavior of coefficients ω is given by β-function (II.5-82). The large-N limit of the $SU(N)$-theories is given by the interaction of $I = 2$ and $I = 5$ subspaces (note that the ω^I have different N order, which has to be taken into account). The crossing matrix, which describes the large-N behavior of $\omega^{2(5)}$, is

$$C_{su}^{\text{Large N}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{III.4-65}$$

This matrix is singular. Therefore, it does not corresponds to any "usual" bosonic theory.

The coefficients $\omega^{I=2}$ ($\omega^{I=5}$) have only even (odd) components in partial wave basis. Combining these coefficients in one, $\omega_{nl}^{Large-N} = \omega_{nl}^{I=2}+\omega_{nl}^{I=5}\Big|_{Large-N}$ we rewrite eqn. (III.2-37) with the matrix (III.4-65) in the form

$$\omega_{nl}^{Large-N} = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\sum_{l'=0}^{n}\frac{\omega_{il'}^{Large-N}\omega_{n-i,l'}^{Large-N}}{2l'+1}\left(\delta^{l'l} + \Omega_n^{l'l}\right), \tag{III.4-66}$$

with the initial value for the iteration $\omega_{1l}^{Large-N} = \{1, 1\}$. Comparing eqn. (III.4-66) with the general exp. (III.2-38) we can guess that the effective theory for $SU(N)$ large-N LLogs is a theory with one-component field. The β-function for such a theory is given by the first two diagrams in fig.II.3. This situation is typical for the four-fermionic interaction where the u-channel diagram is zero due to the anticommutativity of the fields. Therefore, we suppose that the effective theory for $SU(N)$ large-N LLogs would be a theory with the scalar Grassman fields.

The large-N limit of $O(N)$-symmetric theories was given by the interaction of only the $I = 0$ component of the vertices, i.e the large-N limit can be obtained by projecting the graph and every of its vertex on the 0-th subspace and forcing all other vertices to be zero. It corresponds to the switching off all diagrams except for the chain diagrams. For the $SU(N)$-theories one has to keep $I = 2, 5$ components. Note that in the usual notation the projectors $P_{2(5)}^{ab,cd}$ are $d^{abk}d^{kcd}$ ($f^{abk}f^{kcd}$), where

$d^{abc}(f^{abc})$ are (anti)symmetric $SU(N)$ structure constants.

III.5 Discussion

The method of obtaining the LLog coefficients through the analytic properties of the amplitude presented in the current chapter is independent from the RG method discussed in the previous chapter. In comparison with the RG method the "analytical" method has the following advantages: one does not need to perform the loop calculation and the resulting recursive equation (III.2-37) is diagonal in all auxiliary indices. Both these advantages make the usage of "analytical" method preferable in comparison to the RG method. However, the analytical extraction of the LLog seems to be possible only for the physical amplitudes, i.e. Green functions can not be considered within this framework. Also the area of applicability of the presented method is restricted to the ϕ^4-type theories, since the three-particle massless amplitudes have no physical domain of kinematic.

In the thesis only the theories in the even $D \geqslant 4$ dimensional space are considered. However, the presented method allows one to consider the $D = 2$ and odd-D theories as well. This would be a very good application for the method, since the direct calculation of the IR asymptotic behavior of the perturbative expansion in the odd-D spaces is a difficult and still not completely investigated task.

The multi-particle amplitudes can be also considered with the help of the analytical properties. The main difficulty for this consideration is to find an expression for a would-be partial wave decomposition in the multi-particle kinematics and their properties in the complex-s plane. Note, that the unitarity relation is quadratic for any process. Therefore, the recursive equations on the LLog coefficients in the "physical" variables (at least at LLog approximation) will be also quadratic. In that way, the equation for the 6π-scattering amplitude LLog coefficients, considered in the previous chapter (II.6-93), can be expressed in the "quadratic" form, i.e. $R \sim \sum \omega R$, where R is given by exp. (II.6-92). From the RG point of view that means that the β-functions (β and $\tilde{\beta}$ in eqn. (II.6-93)) are strongly related to each other.

The usage of the partial wave basis is not crucial for the discussion. Its preference is in the diagonality of the unitary relation. One can use any convenient decomposition of the amplitude. The example of another decomposition is the double Taylor decomposition over s and t. In this case the coefficients ω are transformed to the coefficients χ by exp. (III.3-61). The resulting recursive equation for χ is not diagonal. Another interesting example of the amplitude decomposition is the cone-function decomposition. In the cone expansion one has the continuous parameter μ instead of the discrete index l, and the unitarity relation is also diagonal [60]. Therefore, we expect the diagonal integral equation for the LLog coefficients for this basis.

IV

Application of LLog summation to the hard processes

The hard processes are a wide class of the reactions which probe the intrinsic structure of the hadrons. These processes are described by the partonic distributions. Using an EFT one can perform the systematic expansion of the partonic distributions over the soft kinematic parameters, such as masses, soft momentum transfer, etc. However, the non-local structure of the matrix element induces the terms with an explicit singularity which order increase along with the expansion order. These singular terms strongly dominate in the low-x region and can result in unexpected singularities in the scattering amplitude. Therefore, one has to resum them at all orders. The resummation can be done with the help of the LLog calculation technique described in the previous chapters.

In the present chapter, we review the basics of the EFT applications to the hard processes. We present the detailed calculation of the singular contributions to the pion PDF and GPD within the framework of the $O(N+1)/O(N)$ σ-model. We show that summation of singular terms produces the smooth function, which describes the dominant at small-x and/or large-$b_\perp$ contribution to PDF and GPD. We present the detailed investigation of the properties and the numerical estimations for the resummed singular part of the parton distributions.

IV.1 Hard processes and ChPT

The hard processes are the wide class of the processes with participation of the hadrons at high energies. The main goal for considering of these processes is the investigation of the hadron structure. There are plenty of reviews and textbooks on

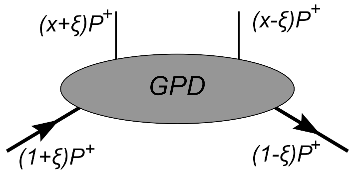

Figure IV-1: The GPD kinematic agreement.

this topic, e.g. [63],[64],[65],[66]. In the present section, we review only some basics which are needed in our consideration. For concreteness we consider the deeply virtual compton scattering (DVCS): $\gamma^* h \to \gamma h$, as a typical example of the hard processes.

In DVCS the role of hard momentum is played by the momentum of the incoming photon: $-q^2 = Q^2 \gg \Lambda^2_{\rm QCD}, M_h^2, ..$. The presence of the large external scale allows one to interpret the process as the scattering of the photon on an individual parton (quark or gluon) from the hadron. This statement can be formulated in the form of the well-known factorization theorem, e.g. [61]. The factorization theorem reads that the amplitude of a hard process can be presented as the convolution of the soft and hard parts,

$$\mathcal{A}(Q^2, P, \mu^2, M^2, ..) = C(Q^2, \mu^2) \otimes O(\mu^2, P, M^2, ..),$$

where C is the coefficient function and O is the soft part. The coefficient function C depends only on the hard scale and can be calculated within the framework of the QCD perturbative expansion. The soft part can not be obtained within the perturbative QCD and has to be parameterized. Both parts depend on the QCD evolution parameter μ^2, and their QCD evolution can be obtained with the help of the renormalization group.

The DVCS amplitude at the leading order has the following form (up to numerical constant)

$$\mathcal{A} = \alpha \int_{-1}^{1} \frac{H(x,\xi,t)}{\xi - x - i\varepsilon} dx + \mathcal{O}(\alpha^2, Q^{-2}), \tag{IV.1-1}$$

where $\xi = \frac{(p-p')_+}{(p+p')_+}$, $p(p')$ is the incoming(outgoing) momentum of the hadron (see fig.4.1), p_+ is its projection on the light-cone direction n, $n^2 = 0$, $t = (p-p')^2$, and $H(x,\xi,t)$ is the generalized parton distribution (GPD) of the hadron.

The GPD is the Fourier transformed matrix element of the non-local light-cone operator

$$H(x,\xi,t) = \int \frac{d\lambda}{2\pi} e^{-ixP\lambda} \langle h(p')|\hat{O}(\lambda)|h(p)\rangle, \tag{IV.1-2}$$

where $P = \frac{1}{2}(p + p')_+$. The twist-2 operator has the following form

$$\hat{O}(\lambda) = \bar{q}\left(\frac{\lambda n}{2}\right) W[\frac{\lambda n}{2}, -\frac{\lambda n}{2}]\gamma^+ q\left(-\frac{\lambda n}{2}\right),$$

where $W[x, y]$ is the Wilson line from x to y.

There are several exceptional values of the GPD arguments:

- The forward limit of GPD is the parton distribution function (PDF),

$$H(x, 0, 0) = q(x). \tag{IV.1-3}$$

  PDF is a more simple object, which appears in many hard reactions, e.g. deep inelastic scattering, Drell-Yan process. PDF is interpreted as the probability to find the parton with momentum fraction x inside the hadron.

- At $\xi = 0$, GPD provides information on the transverse position of partons. The Fourier transform

$$H(x, \vec{b}_\perp) = \int \frac{d^2\Delta}{(2\pi)^2} H(x, 0, -\Delta^2) e^{-i\vec{\Delta}\vec{b}_\perp}, \tag{IV.1-4}$$

  has the meaning of the probability density to find the parton with the given momentum fraction x and with coordinate $\vec{b}$ in the transverse plane.

- The imaginary part of the DVCS amplitude, which is observable by itself, is given by the GPD slice at $x = \xi$, i.e.

$$\mathrm{Im}\mathcal{A}(\xi, t) = H(\xi, \xi, t). \tag{IV.1-5}$$

GPDs differ by quantum numbers and hadronic brackets. In the thesis we discuss the pion GPD and PDF only.

For the application of EFT to parton distribution one has to match the non-local light cone QCD operator to an operator formulated in EFT degrees of freedom. Let us consider left and right twist-2 quark operators on the light cone:

$$\begin{aligned} O^L_{fg}(\lambda) &= \bar{q}_g\left(\frac{\lambda n}{2}\right) \not{n}\, \frac{1+\gamma_5}{2}\, q_f\left(-\frac{\lambda n}{2}\right), \\ O^R_{fg}(\lambda) &= \bar{q}_g\left(\frac{\lambda n}{2}\right) \not{n}\, \frac{1-\gamma_5}{2}\, q_f\left(-\frac{\lambda n}{2}\right). \end{aligned} \tag{IV.1-6}$$

Here, the vector n^μ is the light-cone vector, $n^2 = 0$, and f, g stand for flavour indices. The Wilson line along the straight line between the points $(\lambda n/2)$ and $(-\lambda n/2)$ is assumed. In terms of effective degrees of freedom the operators (IV.1-6) have the form

$$O^{L,R}(\lambda) = F \otimes O^{L,R}_{\text{eff}}(\lambda)\,, \tag{IV.1-7}$$

where $O^{L,R}_{\text{eff}}(\lambda)$ is an effective hadronic operator with the same quantum numbers (but not necessarily with the same twist) as quark operators (IV.1-6) and F is the generating function for the c-number coefficients, which are input for the EFT.

For the description of the pion interactions we will use the two-flavor chiral Lagrangian,

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\,\text{tr}\left[\left(\partial_\mu U \partial_\mu U^\dagger\right) + m_\pi^2\left(U + U^\dagger\right)\right]. \tag{IV.1-8}$$

The m_π and F_π in the chiral Lagrangian (IV.1-8) are the physical values of the pion mass and decay constant. The difference between the physical and bare values of constants is irrelevant for our discussion, since we are going to consider the LLog approximation only.

The $SU(2)$ matrix U in eqn. (IV.1-8) can be parameterized in the form:

$$U = \frac{1}{F_\pi}(\,\sigma + i\,\pi\cdot\tau), \quad \sigma = F_\pi\sqrt{1-\pi^2/F_\pi^2}. \tag{IV.1-9}$$

Substituting parametrization (IV.1-9) into the Lagrangian (IV.1-8) one obtains

$$\mathcal{L}_2 = \frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi_a\partial^\mu\pi^a\right] + \sigma\; F_\pi\; m_\pi^2, \quad \sigma^2 + \sum_{a=1}^{3}\pi^a\pi^a = F_\pi^2. \tag{IV.1-10}$$

In order to obtain larger parametrical freedom we extend this model by increasing the number of the Goldstone bosons up to N. Therefore, the Lagrangian of the model assumes the following form

$$\mathcal{L}_2 = \frac{1}{2}\left[\partial_\mu\sigma\partial_\mu\sigma + \partial_\mu\pi^a\partial_\mu\pi^a\right] + \sigma\; F_\pi\; m_\pi^2, \quad \sigma^2 + \sum_{a=1}^{N}\pi^a\pi_a = F_\pi^2. \tag{IV.1-11}$$

We are going to use this Lagrangian as the EFT for pions. Note, that in the massless limit Lagrangian (IV.1-11) coincides with the Lagrangian of the $O(N+1)/O(N)$ model (II.2-5).

The operators $O^{R,L}$ must be constructed from the Lagrangian buildings blocks, i.e. form the field U, derivatives and masses. The similar procedure have been done for the FF operators (II.4-58) in chapter 2. But for operators of type (IV.1-7), the standard Weinberg counting rules have to be extended. The point is that, although the soft part of the hard processes does not depend on the hard momentum of the QCD factorization, the operator still contains the information about it in the form of the light-cone vector n^μ.

Let us obtain the chiral weight for the components of the vector n. The light-cone decomposition of any four-vector V^μ reads:

$$V^\mu = V^+ \, \widetilde{n}^\mu + V^- \, n^\mu + V_\perp^\mu \,, \tag{IV.1-12}$$

where n^μ and $\widetilde{n}^\mu$ are the light-cone vectors $n^2 = \widetilde{n}^2 = 0$ which are normalized as $n \cdot \widetilde{n} = 1$. These two vectors define a two-dimensional plane. The perpendicular to n and $\widetilde{n}$ plane is called a transverse plane. The vectors from the transverse plane $V_\perp^\mu$ satisfy the condition $n \cdot V_\perp = \widetilde{n} \cdot V_\perp = 0$. The physical observables are obviously invariant under the rescaling of the vector n, i.e. under the transformation $n^\mu \to c\, n^\mu$, where c is an arbitrary non-zero constant. This invariance corresponds to the boost invariance of the physical observables. It is convenient to fix the normalization of the light-cone vector n^μ by the condition $n \cdot p = 1$, where p is one of the small external momenta entering the soft part of the amplitude. Such condition implies that $n^\mu \sim O(1/E)$ and $\widetilde{n}^\mu \sim O(E)$, where E is assumed to be a generic soft momentum as in the usual power counting (I.1-8).

Summing up, we have to construct an effective hadronic operator in eqn. (IV.1-7) using the chiral fields $U(x)$ and their derivatives as building blocks with the following counting rules:

$$n \cdot \partial U(x) \sim O(p^0), \quad \widetilde{n} \cdot \partial U(x) \sim O(p^2), \quad \partial_\perp U(x) \sim O(p). \tag{IV.1-13}$$

Operators (IV.1-6) are matched as follows

$$\begin{aligned} O^L_{fg}(\lambda) &= \frac{iF_\pi^2}{4} \mathcal{F}(\beta,\alpha) * \left[U\left(\frac{\alpha+\beta}{2}\lambda n\right) n\cdot \overleftrightarrow{\partial}\, U^\dagger \left(\frac{\alpha-\beta}{2}\lambda n\right)\right]_{fg} + \ldots, \\ O^R_{fg}(\lambda) &= \frac{iF_\pi^2}{4} \mathcal{F}(\beta,\alpha) * \left[U^\dagger\left(\frac{\alpha+\beta}{2}\lambda n\right) n\cdot \overleftrightarrow{\partial}\, U \left(\frac{\alpha-\beta}{2}\lambda n\right)\right]_{fg} + \ldots\,, \end{aligned} \tag{IV.1-14}$$

where the asterisk denotes the integral convolution with respect to β and α:

$$\mathcal{F}(\beta,\alpha) * O(\beta,\alpha) \equiv \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \; \mathcal{F}(\beta,\alpha)\, O(\beta,\alpha). \tag{IV.1-15}$$

Here $F(\beta,\alpha)$ is the generating function of the tower of the low-energy constants and $\overleftrightarrow{\partial}$ denotes $\overrightarrow{\partial} - \overleftarrow{\partial}$. The low-energy constants characterize the intrinsic structure of the pion and they are not determined within EFT. The ellipsis in equations (IV.1-14) stands for the operators that do not contribute to the one- and two-pion matrix elements of the operators $O^{L,R}$, or which are of the higher orders in the chiral counting. Note that if one will consider the chiral corrections, say, for three-pion distribution amplitudes one need to add additional operators to eqn. (IV.1-14).

At $N=3$, the GPDs of the processes with iso-spin 0 and 1 are defined as [1]:

$$\int \frac{d\lambda}{2\pi} e^{-iP_+ x\lambda} \left\langle \pi^b(p') \left| \mathrm{tr}\left[\tau^c O_{L+R}(\lambda)\right] \right| \pi^a(p) \right\rangle = 4i\varepsilon[abc] H^{I=1}(x,\xi,t), \quad \text{(IV.1-16)}$$

$$\int \frac{d\lambda}{2\pi} e^{-iP_+ x\lambda} \left\langle \pi^b(p') \left| \mathrm{tr}\left[O_{L+R}(\lambda)\right] \right| \pi^a(p) \right\rangle = 4\delta^{ab} H^{I=0}(x,\xi,t). \quad \text{(IV.1-17)}$$

Consequently, the corresponding PDFs are defined as

$$\int \frac{d\lambda}{2\pi} e^{-ip_+ x\lambda} \left\langle \pi^b(p) \left| \mathrm{tr}\left[\tau^c O_{L+R}(\lambda)\right] \right| \pi^a(p) \right\rangle = 4i\varepsilon[abc] q^{I=1}(x), \quad \text{(IV.1-18)}$$

$$\int \frac{d\lambda}{2\pi} e^{-ip_+ x\lambda} \left\langle \pi^b(p) \left| \mathrm{tr}\left[O_{L+R}(\lambda)\right] \right| \pi^a(p) \right\rangle = 4\delta^{ab} q^{I=0}(x)\,. \quad \text{(IV.1-19)}$$

The generalization of the operators for the case of an arbitrary N can be done in the following manner. The isovector operator becomes

$$O^{[ab]}(\lambda) = -\,\mathcal{F}(\beta,\alpha) * P_1^{ab,cd} \pi^c(x_1\lambda n)\, i \overleftrightarrow{\partial}_+ \,\pi^d(x_2\lambda n), \quad \text{(IV.1-20)}$$

where P_1 is the projector on the isospin-1 invariant subspace (III.3-55), and

$$x_1 = \frac{\alpha+\beta}{2}, \quad x_2 = \frac{\alpha-\beta}{2}.$$

Therefore, the PDF at arbitrary N is defined as

$$\int \frac{d\lambda}{2\pi} e^{-ip_+ x\lambda} \left\langle \pi^d(p) \left| O^{[ab]}(\lambda) \right| \pi^c(p) \right\rangle = 8P_1^{cd,ba}\, q^{I=1}(x), \quad \text{(IV.1-21)}$$

and similar for the GPD. The isoscalar PDF (and similar for the GPD) at arbitrary N is defined as

$$\int \frac{d\lambda}{2\pi} e^{-ip_+ x\lambda} \left\langle \pi^b(p) \left| O(\lambda) \right| \pi^a(p) \right\rangle = 2\delta^{ab}\, q^{I=0}(x)\,, \quad \text{(IV.1-22)}$$

where the operator is defined as:

$$O(\lambda) = -\mathcal{F}(\beta,\alpha) * \Big[\sigma(x_1\lambda) i \overleftrightarrow{\partial}_+ \sigma(x_2\lambda) + \pi^a(x_1\lambda) i \overleftrightarrow{\partial}_+ \pi^a(x_2\lambda)\Big]. \quad \text{(IV.1-23)}$$

Computing the matrix elements (IV.1-22) and (IV.1-21) at the tree level, one

[1] Our definition of iso-scalar GPD and PDF differ by factor 2 from the definition of [23],[27], i.e. $Q(x) = 2q^{I=0}(x)$. We use such definition for the universality of notations. The physical meaning of distributions is fixed by the expressions (IV.1-27)-(IV.1-28)

obtains the following expressions for the GPDs at the leading order of ChPT

$$\overset{o}{H}{}^{I}(x,\xi) = \int [d\alpha d\beta]\; F^I(\beta,\alpha)\;\left[\delta(x-\xi\alpha-\beta) - (1-I)\;\xi\delta(x-\xi(\alpha+\beta))\right]. \tag{IV.1-24}$$

Here we introduce the notations: $F^{1,0}(\alpha,\beta) = \frac{1}{2}(\mathcal{F}(\beta,\alpha) \pm \mathcal{F}(-\beta,\alpha))$; the measure $[d\alpha d\beta]$ stands for the integration over the rombus $|\alpha|+|\beta| \leq 1$, (IV.1-15).

In the first term of eqn. (IV.1-24) one recognizes the double distribution (DD) representation for the GPD [67],[68]. The second term of eqn. (IV.1-24), which contributes only to the singlet GPD corresponds to the GPD D-term [69]. Note that the D-term for the pion GPD is also fixed in terms of DD due to the soft pion theorem [70]

$$\overset{o}{H}{}^{I=0}(x,\xi=\pm 1) = 0.$$

The generating function $F(\beta,\alpha)$ for the low-energy chiral constants at the leading order of ChPT coincides with the DD for the pion in the chiral limit and at zero momentum transfer. This implies that the function $F(\beta,\alpha)$ is related to the quark distributions of the pion in the chiral limit ($m_\pi = 0$):

$$\int_{-1+|\beta|}^{1-|\beta|} d\alpha\; F^{I=0}(\beta,\alpha) = \frac{1}{2}\left[\theta(\beta)\,\overset{o}{q}(\beta) - \theta(-\beta)\,\overset{o}{\bar{q}}(-\beta)\right] = q^{I=0}(x), \tag{IV.1-25}$$

$$\int_{-1+|\beta|}^{1-|\beta|} d\alpha\; F^{I=1}(\beta,\alpha) = \theta(\beta)\,\overset{o}{q}(\beta) + \theta(-\beta)\,\overset{o}{\bar{q}}(-\beta) = q^{I=1}(x). \tag{IV.1-26}$$

The first Mellin moment of these distributions is related to the forward matrix elements of the energy momentum tensor and vector current respectively. This gives:

$$\int [d\alpha d\beta]\; F^{I=0}(\beta,\alpha)\beta \;=\; \frac{1}{2}M_2^Q, \tag{IV.1-27}$$

$$\int [d\alpha d\beta]\; F^{I=1}(\beta,\alpha) \;=\; 1\,, \tag{IV.1-28}$$

where we have introduced the notation for the fraction of the pion momentum carried by quarks and antiquarks $M_2^Q = \int_0^1 dx\; x\;(q(x)+\bar{q}(x))$. Using these equations and eqn. (IV.1-24) one can obtain the first moment for GPD $H^{I=1,0}$:

$$\begin{aligned} \int_{-1}^{1} dx\, x\, \overset{o}{H}{}^{I=0}(x,\xi) &= \frac{1-\xi^2}{2}M_2^Q \\ \int_{-1}^{1} dx\; \overset{o}{H}{}^{I=1}(x,\xi) &= 1\,. \end{aligned} \tag{IV.1-29}$$

The expressions with defined C-parity are convenient for the EFT calculation, but for practical applications one needs the expressions for distributions of particular flavor in particular pion. It is straightforward to show that

$$H^u_{++} = \int \frac{d\lambda}{2\pi} e^{-ixP_+\lambda} \langle \pi^+(p')|\bar{u}\left(\frac{\lambda n}{2}\right)(n\gamma)u\left(-\frac{\lambda n}{2}\right)|\pi^+(p)\rangle = H^{I=0} + H^{I=1}, \tag{IV.1-30}$$

$$H^d_{++} = \int \frac{d\lambda}{2\pi} e^{-ixP_+\lambda} \langle \pi^+(p')|\bar{d}\left(\frac{\lambda n}{2}\right)(n\gamma)d\left(-\frac{\lambda n}{2}\right)|\pi^+(p)\rangle = H^{I=0} - H^{I=1}, \tag{IV.1-31}$$

where we omit the arguments of GPD for compactness.

The interpretation of the generating functions $F^I(\beta,\alpha)$ as DD's assumes that these functions depend on the factorization scale μ. The functional dependence from this parameter is described by the evolution equations [72]. For the sake of simplicity we do not write the argument μ explicitly but imply it.

IV.2 Singular contribution to pion distribution functions

The parton distribution in the chiral limit is given by exp. (IV.1-24). This expression corresponds to the tree order diagram shown in fig.IV-2.a, where by the crossed vertex we denote the non-local operator (IV.1-20), (IV.1-23). The leading chiral correction to the parton distribution within the framework of ChPT was considered in [62],[71]. The correction is given by the two diagrams shown in fig.IV-2.b-c. The non-analytic part of the result for the isovector PDF reads

$$q(x) = \overset{o}{q}(x) + a_\chi \ln\left(\frac{1}{a_\chi}\right)\left(\overset{o}{q}(x) - \delta(x)\right), \tag{IV.2-32}$$

where $a_\chi = \frac{m_\pi^2}{(4\pi F_\pi)^2}$.

Exp. (IV.2-32) contains an explicit singularity, namely the δ-function. As we will show later, the higher order chiral corrections contain higher order singularities in form of derivatives of δ-function. This may indicate that in the area of small-x where the δ-functions became large the usual chiral expansion is not valid anymore.

The source of the δ-function in exp. (IV.2-32) is the loop diagram shown in fig.IV-2.c. Let us consider this diagram in details. The Feynman rule for the non-local 2-pion vertex is the following

$$\mathcal{V}^I = iT^I_{ab}F^I(\beta,\alpha)\Big[k_+\delta(xP_+ - \beta k_+ - \alpha\frac{\Delta_+}{2}) - (1-I)\frac{\Delta_+}{2}\delta(xP_+ - (\beta+\alpha)\frac{\Delta_+}{2})\Big],$$

where $T^0_{ab} = \delta^{ab}$ and $T^1_{ab} = \frac{1}{2}P_1^{[cd],ab}$. The loop-integral of the diagram in fig. IV-2.c

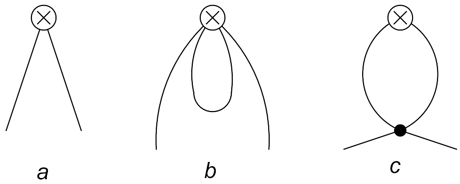

Figure IV-2: a. The tree order of GPD. b,c. The leading chiral corrections to GPD.

has the following form

$$\text{Diag.c} \sim \frac{a_\chi}{m_\pi^2} F(\beta,\alpha) * \int dk \frac{k_+\delta(xP_+ - \beta k_+)\cdot \text{Denom.}}{(k^2-m^2)^2}, \tag{IV.2-33}$$

where Denom. is a linear combination of three momentum structures (kP), k^2 and m^2. Integral (IV.2-33) contains one open Lorentz index, which is contracted with n_μ. The only vector which is acceptable for the contraction with n_μ in PDF kinematics is the momentum P_μ. Therefore, the term proportional to k^2 and m^2 in Denom. produces zero. Expanding the δ-function at $k_+ = 0$ in a formal series

$$\delta(k_+\beta - P_+x) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!}\left(\frac{k_+}{P_+}\right)^n \delta^{(n)}(x), \tag{IV.2-34}$$

we conclude that only the term with $n = 0$ survives, since the Denom. contains only one momentum P_μ. The last remainder loop-integral can be taken in an usual way, with the result (IV.2-32). Note, that for the isoscalar PDF the diagram c. produces zero, due to the convolutions of isospin indices.

The presence of the singular term reorganizes the usual chiral expansion at small values of the momentum fractions $x \sim a_\chi$. The kinematical region of values $x \sim a_\chi$ is equivalent to the large light-cone distance $\lambda \sim 1/a_\chi$. Therefore, the singular term, which is formally of the next-to-leading order, actually is of the same order as the leading term $a_\chi\delta(x) \sim \mathcal{O}(a_\chi^0)$.

The higher orders of ChPT possess even more singular structures – the derivatives of the δ-function, $\delta^{(n)}(x)$. Let us consider the diagram in fig. IV-2.c with a four-vertex from the $\mathcal{L}_{2n}$ part of the ChPT Lagrangian. The vertex contains $\sim (kp)^n$ term in the Denom. of the loop-integral (IV.2-33). Its loop-convolution with the non-local vertex can be easily calculated with the help of the formal expansion (IV.2-34). It reads

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k_+\delta(\beta k_+ - xP_+)(2kP)^n}{(k^2-m^2)^2} \tag{IV.2-35}$$

$$= \frac{i}{\varepsilon(4\pi)^2}\left(\frac{\mu^2}{m^2}\right)^{\varepsilon} \delta^{(n-1)}(x)\beta^{n-1}\frac{(-1)^{n-1}}{(n-1)!}\left(m^2\right)^n + O(\delta^{n-2},\epsilon^0),$$

where under $O(\delta^{n-2})$ we understand less singular and regular contributions. Therefore, the singular contributions appear at all orders of the chiral expansion.

In the GPD kinematics the presence of a non-zero parameter ξ smears $\delta^{(n-1)}(x)$. As a result, one has the $\theta(x < |\xi|)/\xi^n$ term. These contributions also break down the usual hierarchy of the chiral expansion, since such terms are of the order $\sim a_\chi^0$ in the regime $\xi \sim a_\chi$. Moreover, the amplitude (IV.1-1) calculated on the singular terms contains the unphysical pole at $\xi = 0$.

In the GPD kinematics, the loop integral contains two momenta which can carry the "+" indices, Δ_μ and P_μ. However, the highest singular contribution comes from the highest power of momentum P_μ only. Because any momentum Δ_μ in the denominator of the loop integral decreases the singularity, according to the relation $\Delta_+ = 2\xi P_+$,. Therefore, the singular term is produced by $\sim (kp)^n$ item of the denominator, just as in the PDF case. The basic loop integral in the GPD kinematics is

$$\int dk \frac{k_+\delta(k_+\beta + \frac{\Delta_+}{2}\alpha - xp_+)(2kp)^n}{[(k-\Delta/2)^2 - m^2][(k+\Delta/2)^2 - m^2]} = \qquad \text{(IV.2-36)}$$
$$\frac{i}{\varepsilon}\frac{(-1)^{n-1}}{2n!}\frac{\theta(|x| < |\xi|)}{\xi^n}\int_{-1}^{1} d\eta \frac{\left[\partial_\eta^n \eta\delta(\alpha + \eta\beta - x/\xi)\right]}{\Big[R(\eta,t)\Big]^{-n}}\left(\frac{\mu^2}{R(\eta,t)}\right)^{\varepsilon} + \mathcal{O}(\xi^{-n+1}, \epsilon^0),$$

with

$$R(\eta,t) \;=\; m^2 - \frac{t}{4}(1-\eta^2).$$

The loop-integral of (IV.2-35)-(IV.2-36) type can be also produced in multi-loop diagram. Therefore, we have the double expansion over singular terms and over the loops. The loop diagrams produce different powers of logarithms, which are dominate in our kinematic region.

The hierarchy of the expansion is the following. The most singular terms at any order have the same status as the leading regular contribution. Among the singular terms of the same order, there are terms with different powers of logarithms. [2] Therefore, one has to resum the leading singular contributions at the LLog approximation in order to obtain the selfconsistent leading order result. In the PDF case the singular terms have the form of derivatives of delta-function. In the GPD case the singular terms have the form of inverse powers of ξ. The resulting expansion for

[2]This situation is similar to the perturbative expansion in the usual renormalizable theory, where the coupling constant $g \sim \frac{1}{\ln \Lambda^2}$. Therefore, the perturbative series goes only over the inverse powers of logarithms.

PDF has the form

$$q^0(x) = q^{0,reg}(x) + \sum_{n\geq 2,\,\text{even}} a_n^0 \left[a_\chi \ln\left(\frac{1}{a_\chi}\right)\right]^n \delta^{(n-1)}(x) + \mathcal{O}\left(\ln^{-1}\left(\frac{1}{a_\chi}\right), \frac{1}{a_\chi}\right), \quad \text{(IV.2-37)}$$

$$q^1(x) = q^{1,reg}(x) + \sum_{n\geq 1,\,\text{odd}} a_n^1 \left[a_\chi \ln\left(\frac{1}{a_\chi}\right)\right]^n \delta^{(n-1)}(x) + \mathcal{O}\left(\ln^{-1}\left(\frac{1}{a_\chi}\right), \frac{1}{a_\chi}\right). \quad \text{(IV.2-38)}$$

The evenness(oddness) of the derivatives in the expansion follows from the evenness(oddness) of PDF itself, i.e. $q^I(-x) = (-1)^{I+1} q^I(x)$. In Ref. [23] the coefficients $D_{1,2,3}$ were computed performing the direct three-loop calculations with the result:

$$a_1^1 = -1, \quad a_2^0 = -\frac{5}{6}\langle x\rangle, \quad a_3^1 = -\frac{25}{108}\langle x^2\rangle\,, \qquad \text{(IV.2-39)}$$

where

$$\langle x^n\rangle = F^I(\beta,\alpha) * \beta^n = \int_{-1}^{1} dx x^n \overset{o}{q}(x)\,, \qquad \text{(IV.2-40)}$$

the n-th moment of the corresponded PDF.

In the contrast to the previous chapters, in the present chapter we deal with the massive EFT. Therefore, there is no subleading terms in logarithm diagrams, i.e. every diagram contains LLog contribution. However, there is a limited class of diagrams, which can contain the leading singular term. Also, the necessity to extract only the leading singular term from the loop-integrals simplifies the task crucially, namely, the masses in the diagrams can be partly put to zero. Let us discuss these features in details.

The singular contribution is the consequence of the loop convolution of the non-local and local vertices. The power of the singularity is proportional to the number of the vectors P_μ in the denominator, as we have seen in the basic loop integrals (IV.2-35) and (IV.2-36). The diagrams, that are irreducible over the operator vertex (fig.4.3a), have no leading singular contribution, since the clip part (without external legs) does not contain the momentum P_μ. The diagrams with an operator vertex of a higher chiral order have an additional power of the moment k_μ in the integral. Therefore, they are also sub-leading in the singular part. The same situation happens with diagrams with many (more then two) fields in the operator vertex, fig.4.3b.

The only topology of graphs that can produce the leading singular term is shown in fig.IV-3.c. The $\pi\pi$-scattering subgraph of the diagram in fig.IV-3.c can be taken in the massless limit, because any extra power of m^2 decreases the power of the singularity by unity. Therefore, one can use the technique of the previous chapters in order to obtain the LLog behavior the $\pi\pi$-scattering subgraph.

The consideration of the leading singular term is similar to the consideration of

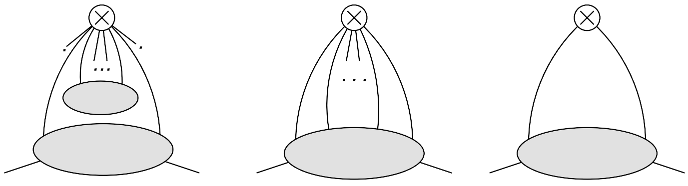

Figure IV-3: Different diagram topologies: a. and b. does not contribute to the leading singular terms.

the FF LLog behavior in the pure massless theory. The main difference is that one does not not have to consider the renormalization of the non-local vertex, since it would decrease the power of the singular term. Schematically, the graph in fig.IV-3.c has the following structure:

$$G = \mathcal{V}(x)\star\Big[((n-1)\text{-loop-subgraph})\ +...+\ g^{(4)}_{nC}V_{nC}\Big]_{\text{massless}}$$
$$= \sum_{k=0}^{n}\ln^n(\mu^2)R_n(x,p,\Delta,m^2),$$

where V_{nC} is the Feynman expression for the n-th order of the four-pion vertex in the EFT (IV.1-11), and the star denotes the loop-convolution. The operator vertex $\mathcal{V}$ is of the lowest chiral order, thus, it has no anomalous dimension. Recalling eqn. (I.4-23), property (II.2-24) and definition (II.3-34) one can perform the chain of transformations:

$$\begin{aligned}\left(\mu^2\frac{d}{d\mu^2}\right)^n G &= n!R_n = \mu^2\frac{d}{d\mu^2}\hat{H}^{n-1}\left(\mathcal{V}(x)\star\Big[...+\sum_C g_{nC}V_{nC}\Big]_{\text{massless}}\right)\\ &= \sum_C\frac{(n-1)!}{F_\pi^{2(n-1)}}\omega_{nC}\mu^2\frac{d}{d\mu^2}\left(\mathcal{V}(x)\star V^{\text{massless}}_{nC}\right),\end{aligned} \qquad \text{(IV.2-41)}$$

where ω_{nC} is defined by eqn. (II.3-34) and satisfy recursive equation (II.3-36).

In the massless limit the theory (IV.1-11) coincides with the earlier considered $O(N+1)/O(N)$ σ-model (II.2-5). Thus, the expression (II.3-42) can be taken for the V_{nC}. The kernel for eqn. (III.2-37) for this theory is given by exp. (II.3-52).

In order to extract the leading singular contribution one has to take the highest power of (kp) from the vertex V_{nC} (II.3-42), contract the indices and calculate the loop convolution with the help of expressions (IV.2-35) and (IV.2-36). The results of the calculation in the PDF and GPD kinematics are presented further.

Table IV-1: First few values of coefficients near the leading singular terms $a_n^{I=0}$(table a) and $a_n^{I=1}$(table b) at arbitrary N.

(a)

n	2	4	6
$a_n^{I=0}$	$\frac{5}{6}-\frac{5N}{6}$	$-\frac{7N^3}{480}+\frac{133N^2}{3240}-\frac{791N}{12960}+\frac{14}{405}$	$-\frac{N^5}{8960}+\frac{361N^4}{708750}-\frac{9441001N^3}{8164800000}+\frac{36581227N^2}{24494400000}-\frac{30246239N}{24494400000}+\frac{2449121}{4898880000}$

(b)

n	1	3	5
$a_n^{I=1}$	-1	$-\frac{N^2}{24}+\frac{37N}{432}-\frac{49}{432}$	$-\frac{N^4}{1920}+\frac{20753N^3}{10368000}-\frac{363091N^2}{93312000}+\frac{17849N}{4665600}-\frac{101}{38880}$

Singular part of pion PDF With the help of eqn. (IV.2-41) and loop-integral (IV.2-35) we obtain the expression for the coefficients a_n^I (IV.2-37)-(IV.2-38):

$$a_n^{I=0} = = \frac{-1}{n!}C_n^{I=0} = \frac{-1}{n!}\Big[\sum_{C=0}^{n}\omega_{nC} + \frac{N}{2}\frac{(2n)!}{n!n!}\omega_{nn}\Big]\,, \quad n = \text{even}, \tag{IV.2-42}$$

$$a_n^{I=1} = = \frac{-1}{n!}C_n^{I=1} = \frac{-1}{n!}\sum_{C=0}^{n}\omega_{nC}\,, \qquad n = \text{odd}. \tag{IV.2-43}$$

The contribution of the singular terms to the PDF denoted by $\delta q^I(x)$ reads

$$\delta q^I(x) = \sum_{n=1}^{\infty}\frac{-C_n^I}{n!}\delta^{(n-1)}(x)\langle x^{n-1}\rangle\left(a_\chi \ln\left(\frac{1}{a_\chi}\right)\right)^n, \tag{IV.2-44}$$

where the moments $\langle x^n\rangle$ are defined by exp. (IV.2-40) and have to be taken over the PDF with the corresponding isospin.

The coefficients C_n^I are proportional to the LLog coefficient of $s = 0$ amplitude with the isospin I:

$$C_n^{I=0} = \frac{1}{2}\sum_{l=0}^{n}\omega_{nJ}^0\Omega_n^{Jl} = \frac{(2n)!}{2(n!)^2}\omega_{nn}^0, \quad n = \text{even}, \tag{IV.2-45}$$

$$C_n^{I=1} = \frac{1}{2}\sum_{l=0}^{n}\omega_{nJ}^1\Omega_n^{Jl} = \frac{(2n)!}{2(n!)^2}\omega_{nn}^1, \quad n = \text{odd}. \tag{IV.2-46}$$

where for the second equality exp. (B.1-7) was used. The first few values of a_n^I are presented in tables IV.1 (a-b).

In order to resum the series of singular terms (IV.2-44) we introduce the Mellin image for the coefficients c_n^I in the following way:

$$\frac{C_n^I}{A^{n-1}n} = \frac{(2n)!}{(n!)^2A^{n-1}}\frac{\omega_{nn}^I}{2n} = \int_0^\infty f^I(z)z^{n-1}\,dz, \tag{IV.2-47}$$

Table IV-2: First few numerical values of coefficients near the leading singular terms $a_n^{I=0,1}$ at $N = 3$.

n	1	3	5	7	9	11
$a_{n+1}^{I=0}$	-1.67	-0.17	-6.85×10^{-3}	-1.47×10^{-4}	-2.01×10^{-6}	-1.90×10^{-8}
$a_n^{I=1}$	$-1.$	-0.23	-0.014	-4.22×10^{-4}	-7.37×10^{-6}	-8.54×10^{-8}

n	13	15	17	19
$a_{n+1}^{I=0}$	-1.32×10^{-10}	-7.03×10^{-13}	-2.95×10^{-15}	-1.01×10^{-17}
$a_n^{I=1}$	-7.05×10^{-10}	$-4.35\times^{-12}$	-2.08×10^{-14}	-7.94×10^{-17}

where A is the number which specifies the asymptotic behavior of ω_n, see (IV.2-51). Representing the n-th derivative of the delta-function by the expression:

$$\delta^{(n)}(x) = \int \frac{d\lambda}{2\pi}(i\lambda)^n e^{i\lambda x}, \tag{IV.2-48}$$

changing the order of summation and integration we rewrite the sum (IV.2-44) in the form

$$\delta q^I(x) = -\frac{\epsilon}{A}\int_{-1}^{1} d\beta \int_0^\infty dz \int \frac{d\lambda}{2\pi} \overset{o}{q}{}^I(\beta) f^I(z) \sum_{n=1}^{\infty} \frac{1+(-1)^{I+n}}{2}\frac{(i\lambda\epsilon z\beta)^{n-1}}{(n-1)!} e^{i\lambda x},$$

where

$$\epsilon = A\ a_\chi \ln\left(\frac{1}{a_\chi}\right).$$

Computing the sum and integrating over λ one obtains

$$\delta q^I(x) \quad = \quad -\frac{\epsilon}{2A}\int_{-1}^{1} d\beta \int_0^\infty dz\ \overset{o}{q}{}^I(\beta) f^I(z)\left(\delta(x+\epsilon z\beta) - (-1)^I \delta(x-\epsilon z\beta)\right). \tag{IV.2-49}$$

Using the symmetry properties of the PDF, i.e. $q^I(-x) = (-1)^{1-I} q(x)$ one rewrites exp. (IV.2-49) in the form of Mellin convolution:

$$\begin{aligned}\delta q^I(x) \quad &= \quad \frac{(-1)^I\,(\mathrm{sign}(x))^{I-1}}{A}\left[\overset{o}{q}{}^I \otimes f^I\right]\left(\frac{|x|}{\epsilon}\right) \\ &= \frac{(-1)^I\,(\mathrm{sign}(x))^{I-1}}{A}\int_0^1 \frac{d\beta}{\beta}\ \overset{o}{q}{}^I(\beta) f^I\left(\frac{|x|}{\epsilon\beta}\right).\end{aligned} \tag{IV.2-50}$$

The function δq^I is a nonsingular on the range $-1 < x < 1$ function, if the function $f(z) \sim z^0$ at $z = 0$. Therefore, the resummation solves the problem of the singular contributions

The function $f(z)$ (IV.2-47) can not be found exactly, since the exact analytical solution of eqn. (II.3-36) is not found yet. Instead of it we are going to use the approximate fit of the solution, that is based on the asymptotic behavior of the solution and on the large-N expansion.

We assume that the asymptotic behavior of the ω_n at $n \to \infty$ is power-like

$$\omega_{n0} \sim \sum_{l=0}^{n} \omega_{nl} \sim A^{n-1}, \tag{IV.2-51}$$

where A is a number. Such behavior is natural for the large class of the recursive equations, see the more detailed discursion in appendix C.2. Also such behavior is supported by the large-N expansion (II.3-55) and by the numerical investigations. The higher partial waves coefficients have a similar or weaker behavior. Particulary, the crossing symmetry implies that

$$\omega_{nn} \sim \frac{n!n!}{(2n)!}\frac{\omega_{n0}}{n+1} \sim \frac{A^{n-1}}{n+1},$$

which is also supported by the Large-N expansion (C.1-7). Therefore, we suggest the following ansatz with two free parameters for the coefficients a_n^I

$$C_n^I = (A^I)^{n-1}\left(c_0^I + \frac{c_1^I}{n+1}\right), \tag{IV.2-52}$$

where the parameter c_0^I fixes the correct lowest n value.

The numerical calculation of the first 150 coefficients a_n^I at $N = 3$ gives the following numbers for the parameters:

$$A^0 = 1.14861, \quad , c_0^0 = 0.75949, \quad c_1^0 = 2.07464 \ , \tag{IV.2-53}$$

$$A^1 = 1.14861, \quad , c_0^1 = 0.75942, \quad c_1^1 = 0.48116 \ . \tag{IV.2-54}$$

The comparison of the fit with the exact solution is shown in fig.IV-4. One can see that for $n > 40$ the deviation form the exact solution is less then 0.2%. The maximum deviation from the exact solution is reached 15% at points $n \sim 4$.

The parameters A^0 and A^1 coincide, although they are estimated independently. This indicates that our assumption on the asymptotic behavior (IV.2-51) is correct.

Using the fit (IV.2-52) and exp. (IV.2-47), one obtains the approximate function $f(z)$:

$$f_{\text{app}}(z) = \theta(z<1)\left(c_0^I + c_1^I - c_1^I z\right). \tag{IV.2-55}$$

The leading singular correction to the PDF within this approximation is

$$\delta q_{\text{app}}^I = \frac{\theta(|x|<\epsilon)(-1)^I\left(\text{sign}(x)\right)^{I-1}}{A}\int_{|x|/\epsilon}^{1}\frac{d\beta}{\beta}\overset{oI}{q}(\beta)\left(c_0^I + c_1^I - c_1^I\frac{|x|}{\epsilon\beta}\right) \tag{IV.2-56}$$

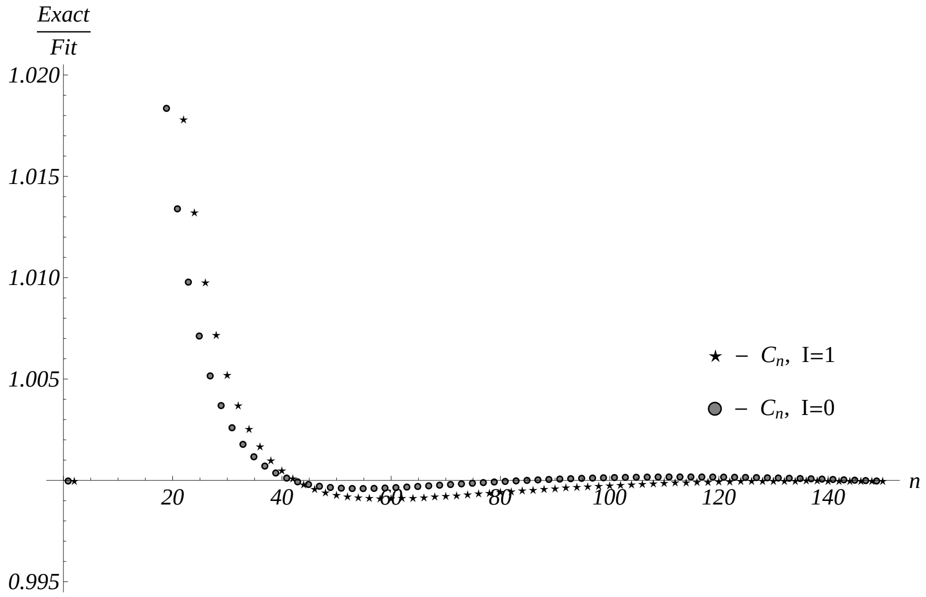

Figure IV-4: The comparison of the exact calculated singular terms coefficient with the fit (IV.2-52).

In the large-N limit the values of the fit parameters (IV.2-53)-(IV.2-54) are following:

$$a^0 = a^1 = \frac{N}{2}, \quad c_0^0 = c_0^1 = 1, \quad c_1^0 = 2, \quad c_1^1 = 0. \tag{IV.2-57}$$

Substituting these values of parameters to exp. (IV.2-56) one obtains the result of ref. [27].

The domain of argument for the function $f(z)$ is most likely restricted to the area $0 < z < 1$, because the Mellin moment of $f(z)$ is not an increasing function of n, i.e. $\frac{C_n}{A^{n-1}} \sim 1$ at $n \to \infty$. Therefore, the domain of variable x is restricted to the area $|x| < \epsilon \sim 0.066$, e.g. see exp. (IV.2-56).

The popular small-x behavior of the PDF is given by the Regge-like ansatz

$$\overset{o}{q}{}^{I}(x) \sim \frac{1}{x^{\alpha_\pm}}, \tag{IV.2-58}$$

where $\alpha_\pm = \alpha_\pm(0)$ is the intercept for the Regge trajectory, $\alpha_- \approx 0.5$, $\alpha_+ \approx 1.1$. The subscript $\pm$ stays for the C-parity of the Regge trajectory. The $\alpha_{+(-)}$ corresponds to the $I = 0(1)$ PDF. Substituting the Regge ansatz to exp. (IV.2-50) and using the properties of the Mellin convolutions one obtains that at small-x the PDF singular part behaves as

$$\delta q^I(x) = (-1)^I\,(\mathrm{sign}(x))^{I-1}\left(\frac{a_\chi \ln(1/a_\chi)}{x}\right)^{\alpha_\pm}\frac{C^I_{\alpha_\pm}}{\alpha_\pm}\left(1 + \mathcal{O}\left(\frac{x}{\epsilon}\right)\right), \tag{IV.2-59}$$

where C^I_α is the analytical continuation of the coefficients (IV.2-42)-(IV.2-43) to the non-integer values of n, which can be done with the help of exp. (IV.2-52). Therefore, the leading singular correction does not change the small-x asymptotic behavior of the PDF.

Interestingly, the leading power of chiral coupling is determined by the intercept of the Regge trajectory α. This shows clearly the importance of the resummation of the singular chiral corrections for the derivation of the leading chiral counting of PDF. The naive chiral counting, i.e. without taking into account the second scale related to the light-cone distance, suggests that the leading chiral correction to the PDF is $\sim a_\chi \ln(1/a_\chi)$. However, it is not correct as it is demonstrated by exp. (IV.2-59).

Singular part of pionic GPD The consideration of the singular terms in the off-forward kinematic is more involved. Using eqn. (IV.2-41) and the loop-integral (IV.2-36) one obtains the following expression for the leading singular part of GPD:

$$\delta H^I(x,\xi,t) = \tag{IV.2-60}$$
$$\frac{-1}{2}\sum_n \frac{1}{\xi^n} C^I_n \int_{-1}^{1} d\eta \frac{(R\ln 1/R\,)^n}{n\; n!} \partial^n_\eta \eta F^I(\alpha,\beta) * \delta\left(\beta\eta+\alpha-\frac{x}{\xi}\right),$$

where

$$R = \frac{m_\pi^2 - t(1-\eta^2)/4}{(4\pi F_\pi)^2},$$

and coefficients C^I_n are given by exp. (IV.2-42)-(IV.2-43).

In the forward limit $\xi, t \to 0$ exp. (IV.2-60) coincides with exp. (IV.2-44), because

$$\lim_{\xi\to 0} \frac{\theta(|x|<\xi)}{\xi^n} \partial^n_\eta \left[\eta\delta\left(\beta\eta+\alpha-\frac{x}{\xi}\right)\right] = n\delta^{(n-1)}(x)\beta^{n-1}, \tag{IV.2-61}$$

and $R|_{t=0} = a_\chi$.

The summation in exp. (IV.2-60) can be done in the similar way as for the PDF. Using the representations (IV.2-47)-(IV.2-48) one obtains

$$\delta H^I(x,\xi,t) = \frac{-1}{4A}\int_{-1}^{1} d\eta F^I(\alpha,\beta) * \int_0^\infty dz f^I(z) \int \frac{d\lambda}{2\pi} e^{i\lambda\left(\beta\eta+\alpha-\frac{x}{\xi}\right)}$$
$$\times \left[\left(\frac{\eta}{z}+\frac{\epsilon_t}{\xi}\right) e^{i\lambda\frac{\beta\epsilon_t z}{\xi}} + (-1)^I \left(\frac{\eta}{z}-\frac{\epsilon_t}{\xi}\right) e^{-i\lambda\frac{\beta\epsilon_t z}{\xi}} - (1+(-1)^I)\frac{\eta}{z}\right],$$

where $\epsilon_t = AR\ln\left(\frac{1}{R}\right)$. Integrating over λ, and rescaling the variable z one obtains the expression similar to exp. (IV.2-50):

$$\delta H^I(x,\xi,t) = \frac{-1}{2A}\int_{-1}^{1} d\eta \int_0^\infty \frac{dz}{z} f^I(\frac{z}{\epsilon_t})\ h^I(x,\xi,z,\eta), \tag{IV.2-62}$$

where f^I is defined in (IV.2-47) and

$$\begin{aligned} h^I(x,\xi,z,\eta) &= \frac{1}{2}F^I(\beta,\alpha) * \left(D(\eta\xi+z) + (-1)^I D(\eta\xi - z) - (1+(-1)^I)D(\eta\xi)\right), \\ D(y) &= y\delta(\beta y + \xi\alpha - x). \end{aligned} \tag{IV.2-63}$$

Depending on the domain to which the parameters ξ, x and y belong the convolution $F * D(y)$ is represented by the GPD or by the distribution amplitude (DA):

$$\begin{aligned} \overset{o}{H}{}^I(x,\xi) &= F^I * \left(\delta(x-\xi\alpha-\beta) - (1+(-1)^I)\xi\delta(x-\xi\alpha-\xi\beta)\right), \\ \overset{o}{\Phi}{}^I(x,\xi) &= F^I * \left(\xi\delta(x-\alpha-\xi\beta) - (1+(-1)^I)\delta(x-\alpha-\beta)\right). \end{aligned}$$

Therefore, the function h^I (IV.2-63) can be rewriten in the terms of the physical quantities

$$\begin{aligned} h^I(x,\xi,z,\eta) = \theta(x<\eta\xi+z)\theta(\xi<\eta\xi+z)\ \overset{o}{H}{}^I\left(\frac{x}{\eta\xi+z},\frac{\xi}{\eta\xi+z}\right) \\ +\theta(x<\xi)\theta(\eta\xi+z<\xi)\ \overset{o}{\Phi}{}^I\left(\frac{x}{\xi},\frac{\eta\xi+z}{\xi}\right) - \delta_{I0}\theta(x<\xi)\ \overset{o}{\Phi}{}^I\left(\frac{x}{\xi},\eta\right), \end{aligned} \tag{IV.2-64}$$

where we have used the relation $F^I(\beta,\alpha) = (-1)^{I+1}F^I(-\beta,\alpha)$.

The character feature of the singular term resummation is that although every term of the series (IV.2-60) is restricted to the area ($x<\xi$), the resulting expression (IV.2-64) covers a wider region. The singular terms influence on both, the DGLAP and the ERBL, regions of GPD.

The PDF with the transverse momentum dependence $q(x,\Delta^2)$ takes a special place in particle physics. This object is much simpler than the GPD and can be obtained from the later taking the limit $\xi\to 0$, $t=-\Delta^2$.

The limit $\xi\to 0$ can be taken both using exp. (IV.2-61) before the summation and directly in exp. (IV.2-64) with the same result:

$$\delta q^I(x,\Delta^2) = \delta H^I(x,0,-\Delta^2) = \frac{-1}{2A}\int_{-1}^{1} d\eta \int_0^\infty \frac{d\beta}{\beta}\ \overset{o}{q}{}^I(\beta) f^I\left(\frac{x}{\epsilon_t\beta}\right). \tag{IV.2-65}$$

It is easy to see that at $\Delta^2 = 0$ exp. (IV.2-65) coincides with exp. (IV.2-59).

Let us also consider the intersting limit $m_\pi\to 0$. Kinematically this case corresponds to a wide range of the momentum transfer $m_\pi^2 \ll t \ll (4\pi F_\pi)^2$. Using the properties of the Mellin transformation and the integral

$$\int_{-1}^{1} d\eta \left(\frac{1-\eta^2}{4}\right)^n = 2\frac{n!n!}{(2n+1)!},$$

we rewrite exp. (IV.2-65) as

$$\delta q^I_{m=0}(x,\Delta^2) = \frac{-2}{A}\int_0^\infty \frac{d\beta}{\beta} \overset{o}{q}{}^I(\beta) g^I\left(\frac{x}{\epsilon_\Delta \beta}\right), \tag{IV.2-66}$$

where

$$\epsilon_\Delta = 4A\frac{\Delta^2}{(4\pi F)^2}\ln\left(1\Big/\frac{\Delta^2}{(4\pi F)^2}\right),$$

$$\int_0^\infty g^I(z)z^{n-1}\,dz \;=\; \frac{\sqrt{\pi}}{(4A)^{n-1}}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\frac{C^I_n}{n} = \frac{4}{n(2n+1)}\frac{\omega^I_{nn}}{A^{n-1}}. \tag{IV.2-67}$$

Taking the ansatz (IV.2-52) with the values of parameters in the Large-N approximation (IV.2-57) one obtains the result of [27].

The transverse size of the pion cloud The Fourier transformation of the PDF with the transverse momentum dependence gives the impact parameter distribution of partons [73]:

$$q(x,b^2) = \int \frac{d^2\Delta}{(2\pi)^2} q(x,\Delta^2)e^{i(b\cdot\Delta)}. \tag{IV.2-68}$$

The impact parameter distribution $q(x,b^2)$ gives the probability density to find a parton with a momentum fraction x at distance b^2 from the hadron centrum of mass.

The large-b behavior of $q(x,b^2)$ is related to the small-Δ behavior of $q(x,\Delta)$. The later is given by the chiral expansion and its leading term is given by all order resummation of the singular terms (IV.2-65). Note, that the lowest order contribution $\overset{o}{q}(x)$ does not depend on Δ, therefore, it gives the trivial b dependence.

The operation of summation over singular terms and the Fourier transformation do not commute. The reason of the non-commutation is the following. The sum (IV.2-60) is defined for the low-Δ^2 area, thus, for $R\ln(1/R) > 0$. The integration over Δ extends the area $R\ln(1/R) < 0$ also. The analytic continuation of the sum (IV.2-60) to this area contains the addition term, namely, the residue of the point $n = \infty$, which is significant at $R\ln(1/R) = 0$. The accounting of this term is irrelevant for the chiral expansion, because it corresponds to the $\Delta^2 \sim (4\pi F)^2$ region. Therefore, we should take the Fourier transformation and the large-b behavior before the operation of the summation.

The leading order chiral term of $q(x,\Delta^2)$ can be obtained from the complete expression for GPD (IV.2-60) with the help of exp. (IV.2-61). It reads

$$\delta q^I(x,\Delta^2) = \delta H^I(x,0,t=-\Delta_\perp^2) = \frac{-1}{2}\sum_{n=1}^{\infty}\frac{C_n^I}{n!}\int_{-1}^{1} d\eta \left[R\ln\left(\frac{1}{R}\right)\right]^n \langle x^{n-1}\rangle \delta^{(n-1)}(x). \tag{IV.2-69}$$

In order to perform the Fourier transformation we rewrite the logarithms as a differentiation with respect to power:

$$\left(R\ln\left(\frac{1}{R}\right)\right)^n = \left(\frac{d}{d\nu}\right)^n R^{n-\nu}\Big|_{\nu=0}.$$

The Fourier transformation can be done in the usual way

$$\left(\frac{1-\eta^2}{2}\right)^{n-\nu}\int\frac{d^2\Delta}{(2\pi)^2}\,e^{ib\Delta}\left[\frac{4m^2}{1-\eta^2}+\Delta^2\right]^{n-\nu} = \frac{2^{\nu-n}}{\pi\Gamma(-n+\nu)}\frac{(1-\eta^2)^{n-\nu}}{(b^2)^{1+n-\nu}}\left[\alpha^{1+n-\nu}K_{1+n-\nu}(\alpha)\right], \tag{IV.2-70}$$

where $\alpha = \sqrt{\frac{4m^2b^2}{1-\eta^2}}$, $K_n(x)$ is the modified Bessel function of the second order and we have put $(4\pi F)^2 = 1$ for simplicity. Here the integral (IV.2-70) is UV divergent. Therefore, we have calculated it with the regularization factor $e^{-\epsilon\Delta}$ with the subsequent limit $\epsilon\to+0$.

The chiral counting rules assume that $m_\pi^2 \sim \Delta^2 \sim b^{-2}$, and thus, $m^2b^2 \sim 1$. Therefore, the differentiation of the part in the square brackets (IV.2-70) is of the next-to-leading order,

$$\frac{d}{d\nu}\left[\alpha^{1+n-\nu}K_{1+n-\nu}(\alpha)\right] \sim \ln(m^2b^2) = \mathcal{O}\left(\frac{1}{\ln(b^2)}\right).$$

We obtain

$$\left(\frac{d}{d\nu}\right)^n\frac{1}{\pi b^2}\frac{1}{\Gamma(-n+\nu)}\left(\frac{1-\eta^2}{2b^2}\right)^{n-\nu}\left[\alpha^{1+n-\nu}K_{1+n-\nu}(\alpha)\right]\Big|_{\nu=0} = \frac{(-1)^n\,n\,n!}{\pi b^2}\left(\frac{1-\eta^2}{2b^2}\right)^n\ln^{n-1}\left(b^2\right)\left[\alpha^{1+n}K_{1+n}(\alpha)\right]\left(1+\mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right). \tag{IV.2-71}$$

Finally, we obtain that the asymptotic behavior of $q(x,\Delta^2)$ is given by the sum

$$\delta q^I(x,b^2) = \frac{(-1)^I}{\pi(4\pi F)^2b^4}\sum_{n=1}^{\infty}\int_{-1}^{1}\frac{1-\eta^2}{4}\,d\eta\; nC_n^I\,\epsilon_\eta^{n-1}\langle x^{n-1}\rangle\delta^{(n-1)}(x)\left[\alpha^{1+n}K_{1+n}(\alpha)\right]\left(1+\mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right), \tag{IV.2-72}$$

where $\epsilon_\eta = \frac{1-\eta^2}{2(4\pi F)^2b^2}\ln\left((4\pi F)^2b^2\right) > 0$. The summation can be performed using the standard methods. The result is the following

$$\delta q^I(x,b^2)=\frac{1}{\pi A b^2\ln\left((4\pi F)^2 b^2\right)} \tag{IV.2-73}$$
$$\int_{-1}^{1}d\eta\int_0^1\frac{dz}{z}\,[q^I\otimes\tilde f^I](z)\sqrt{\frac{\frac{x}{\epsilon_b z}+m^2b^2}{1-\eta^2}}K_1\left(2\frac{\sqrt{\frac{x}{\epsilon_b z}+m^2b^2}}{1-\eta^2}\right)\left(1+\mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right),$$

where

$$\epsilon_b=\frac{A}{(4\pi F)^2b^2}\ln\left((4\pi F)^2b^2\right),$$
$$\frac{nC_n^I}{A^{n-1}}=\int_0^\infty\tilde f^I(z)z^{n-1}\,dz,$$

and the operation $\otimes$ is the Mellin convolution, see exp. (IV.2-50). Integrating over η we rewrite exp. (IV.2-73) in the form

$$\delta q(x,b^2)=\frac{1}{2\pi A b^2\ln\left((4\pi F)^2b^2\right)} \tag{IV.2-74}$$
$$\int_0^1\frac{dz}{z}[q^I\otimes\tilde f^I](z)\left[yK_0\left(2\sqrt{y}\right)\otimes\sqrt{1-y}_+\right]\left(\frac{x}{\epsilon_b z}+m^2b^2\right)\cdot\left(1+\mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right),$$

where

$$\sqrt{1-y}_+=\sqrt{1-y}\,\theta(|y|<1).$$

Exp. (IV.2-74) is cumbersome. In order to concentrate on the effects of the summation, let us put the mass of pion to zero. In this case the PDF impact parameter dependence has a simple form

$$\delta q^I(x,b^2)=\frac{1}{\pi A b^2\ln\left(b^2(4\pi F)^2\right)} \tag{IV.2-75}$$
$$\int_0^1\frac{dz}{z}\,2\left(\frac{x}{\epsilon_4 z}\right)^{\frac{\nu}{2}}K_\nu\left(2\sqrt{\frac{x}{\epsilon_4 z}}\right)[q^I\otimes\tilde g_\nu^I](z)\left(1+\mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right),$$

where $\epsilon_4=4\epsilon_b$, ν is an arbitrary number $\nu\geqslant 1$ and

$$\frac{nC_n^I}{A^{n-1}}\frac{n!n!}{(2n+1)!}\frac{\Gamma(n+1)}{\Gamma(n+\nu)}=\int_0^\infty\tilde g_\nu^I(z)z^{n-1}\,dz. \tag{IV.2-76}$$

Exp. (IV.2-75) is in fact independent of the parameter ν. The parameter ν is introduced for the purpose of improving the properties of the function $\tilde g_\nu^I$ (IV.2-76).

The integration range of z in exp. (IV.2-75) is naturally separated into two parts, $1>z>\frac{x}{\epsilon_4}$ and $\frac{x}{\epsilon_4}>z>0$. In the area $\frac{x}{\epsilon_4}>z>0$ the Bessel function is exponentially small. Therefore, the main part of the integral (IV.2-75) is concentrated in the area $1>z>\frac{x}{\epsilon_4}$. From this follows that the large-b behavior is exponentially suppressed in the region of $x>\epsilon_4$. That is very natural, because the partons with the high momentum fraction x forms the center of mass of the hadron.

In the low-x region a PDF can be approximated by the Regge-like expression (IV.2-58). The resulting expression is

$$\delta q^I(x,b^2)=\frac{C^I_{\alpha_\pm}}{x^{\alpha_\pm}}\frac{4^{\alpha_\pm}}{\pi}\frac{\Gamma^4(\alpha_\pm+1)}{\Gamma(2\alpha_\pm+2)}\frac{\ln^{\alpha_\pm-1}\left(b^2(4\pi F)^2\right)}{b^2\left(b^2(4\pi F)^2\right)^{\alpha_\pm}}\left(1+\mathcal{O}\left(\frac{x}{\epsilon}\right)\right). \quad \text{(IV.2-77)}$$

We observe a very interesting phenomenon – the distributions of partons in the transverse plane at large impact parameter b depend on the intercept of the corresponding Regge trajectory. This new phenomenon is revealed due to the all order resummation of singular terms.

The large-b behavior can be obtained by the Fourier transformation of the vector and tensor FF, for $I=1$ and $I=0$ respectively. Such consideration gives the following asymptotic behavior

$$q^{I=0}(x,b^2)\sim\frac{\ln(b^2)}{b^6}\delta'(x), \quad q^{I=1}(x,b^2)\sim\frac{1}{b^4}\delta(x). \quad \text{(IV.2-78)}$$

The intercepts of the Regge trajectories are $\omega_-\approx 0.5$ and $\omega_+\approx 1.1$. We see that the distribution of partons at large-b (IV.2-77) drops slower than the naive results (IV.2-78) of the finite order. This comparison clearly shows the importance of the summation of the singular terms.

Note, that eqn. (IV.2-77) can not be obtained directly from exp. (IV.2-59), because of the non-commutativity of the summation over singular terms and Fourier transformation. The root cut $\Delta^2>(4\pi F)^2$ of exp. (IV.2-59) produces a contribution with the higher power of logarithm. This contribution, although it has stronger asymptotic behavior, is irrelevant, because the summation of the series (IV.2-69) is performed in the $\Delta\ll(4\pi F)^2$ region. This mistake has been done in [27].

The behavior of the PDF at $x\ll\epsilon$ can be easily obtained for the massive case as well. It reads

$$\begin{aligned}\delta q^I(x,b^2) \;&=\; \frac{C^I_{\alpha_\pm}}{x^{\alpha_\pm}}\frac{4^{\alpha_\pm}}{\pi}\frac{\Gamma^4(\alpha_\pm+1)}{\Gamma(2\alpha_\pm+2)}\frac{\ln^{\alpha_\pm-1}\left(b^2(4\pi F)^2\right)}{b^2\left(b^2(4\pi F)^2\right)^{\alpha_\pm}} \\ &\quad {}_1F_2\left(-\frac{1}{2}-\alpha_\pm;-\alpha_\pm,-\alpha_\pm;m^2b^2\right)\left(1+\mathcal{O}\left(\frac{x}{\epsilon}\right)\right).\end{aligned} \quad \text{(IV.2-79)}$$

At the large b and fixed m the hypergeometric function behaves as ${}_1F_2(..)\sim e^{-2mb}$.

The function $\tilde g^I$ in exp. (IV.2-75) for practical applications can be approximated with the help of the approximate solution (IV.2-52). Using the arbitrariness of parameter ν one obtains

$$\delta q^I_{\rm app}(x,b^2) = \frac{1}{\pi A}\frac{1}{b^2\ln\left(b^2(4\pi F)^2\right)}\int_0^1 \frac{dz}{z}\left[q\otimes P\right](z) \left(c_0^I\left(\frac{x}{\epsilon_b z}\right)K_2\left(2\sqrt{\frac{x}{\epsilon_b z}}\right)+c_1^I\left(\frac{x}{\epsilon_b z}\right)^{\frac{1}{2}}K_1\left(2\sqrt{\frac{x}{\epsilon_b z}}\right)\right), \tag{IV.2-80}$$

where

$$P(z) = \frac{z}{\sqrt{1-z}} - 2z\ln\left(\frac{1+\sqrt{1-z}}{\sqrt{z}}\right),$$

and the parameters A and $c^I_{0,1}$ are defined in (IV.2-53)-(IV.2-54).

Singular terms resummation for u-quark distribution In the end of the chapter, let us discuss the spatial image of u-quark distribution in π^+. Although, the operator of H^u_{++} has a simple relation with H^I operators, see eqn. (IV.1-30). The resumed results have no such simple relations, because during the above resummation we had assumed the fixed parity properties and summed up even or odd terms only. Here, we do not present the detailed calculation for u-quark distribution, since it is very similar to above, but only the results.

The PDF with the transverse momentum dependence has the following form

$$\delta q^u(x,\Delta^2) = \delta H^u_{++}(x,0,-\Delta^2_\perp) = \frac{-1}{2}\sum_{n=1}^{\infty}\frac{C^u_n}{n!}\int_{-1}^{1}d\eta\left[R\ln\left(\frac{1}{R}\right)\right]^n\langle x^{n-1}\rangle\delta^{(n-1)}(x). \tag{IV.2-81}$$

where the summation goes over all n. The coefficients C^u_n are

$$C^u_n = \sum_{l=0}^{n}\omega_{nl} + \frac{N}{2}\omega_{nn}\frac{(2n)!}{n!n!}. \tag{IV.2-82}$$

The coefficients C^u_n contain both n-even and n-odd parts. These parts have to be considered separately, because their analytical properties differ from each other. Let us introduce the following notation

$$\frac{C^u_n\Big|_{\text{without }(-1)^n\text{ terms}}}{A^{n-1}n} = \int_0^\infty f_+(z)z^{n-1}dz, \tag{IV.2-83}$$

$$\frac{(-1)^n C^u_n\Big|_{\text{only }(-1)^n\text{ terms}}}{A^{n-1}n} = \int_0^\infty f_-(z)z^{n-1}dz. \tag{IV.2-84}$$

With such a separation the summation of series (IV.2-81) gives

$$\delta q^u\left(x,\Delta^2\right) = \frac{-1}{2A}\int_{-1}^{1}d\eta\int_0^1\frac{d\beta}{\beta}\left\{\theta(x>0)\left[f_+\left(\frac{|x|}{\beta\epsilon_t}\right)q^u(-\beta)-f_-\left(\frac{|x|}{\beta\epsilon_t}\right)q^u(\beta)\right]\right.$$
$$\left.+\,\theta(x<0)\left[f_+\left(\frac{|x|}{\beta\epsilon_t}\right)q^u(\beta)-f_-\left(\frac{|x|}{\beta\epsilon_t}\right)q^u(-\beta)\right]\right\}, \qquad \text{(IV.2-85)}$$

where $q^u(\beta) = q_v^u(\beta) + q_s^u(\beta)$ and $q^u(-\beta) = -q_s^u(\beta)$. One can see that this result differs from eqn. (IV.2-65) only by the form of the convolution $q(\beta)$ with $f_\pm$.

The results for the impact parameter dependent PDF are similar to the previous ones, (IV.2-73) and (IV.2-75). The only change is that the convolution in that formulae has to be replaced by

$$[q\otimes f](z) = \int_0^1\frac{d\beta}{\beta}\theta(x>0)\left[q^u(\beta)f_+\left(\frac{z}{\beta}\right)-q^u(-\beta)f_-\left(\frac{z}{\beta}\right)\right]$$
$$+\theta(x<0)\left[q^u(-\beta)f_+\left(\frac{z}{\beta}\right)-q^u(\beta)f_-\left(\frac{z}{\beta}\right)\right], \qquad \text{(IV.2-86)}$$

where $\tilde{g}_\pm$ and $\tilde{f}_\pm$ are defined as in (IV.2-83)-(IV.2-84).

The coefficients (IV.2-82) can be approximated in the form

$$C_n^u = A^{n-1}\left(c_0+\frac{c_1}{n+1}+\frac{(-1)^n c_2}{n+1}\right), \qquad \text{(IV.2-87)}$$

where

$$A = 1.14861,\quad c_0 = 0.7594,\quad c_1 = 1.2451,\quad c_2 = 0.7639.$$

This approximation works as good as the approximations (IV.2-52). In the large-N limit these coefficients are

$$A = \frac{N}{2},\quad c_0 = c_1 = c_2 = 1.$$

The transverse size of the pion grows with smaller x as $1/x^\alpha$, see. exp.(IV.2-77). However, it is interesting to mark, that the full expression (IV.2-73) has two different areas of behavior. In the area of asymptotically small $x \ll \frac{m}{F}$, the size of pion grows asymptotically, i.e. as $1/x^\alpha$. But at $x \sim \frac{m}{F}$ it grows as $\sim x^{-1}$, i.e. stronger. Therefore, this area mainly contribute to the average size of pion, $\langle b^2\rangle$. In the massless limit this "chiral inflation" of the pion leads to the divergence of pion radius, $\langle b^2\rangle \sim \ln m_\pi$. These two areas are clearly seen[3] in the fig.(IV-5).

The popular way to parameterize the Δ-dependence of PDF is so-called "Gribov

[3] As the PDF in the chiral limit $\overset{o}{q}(x)$ we have taken the usual pion PDF, with non-zero mass from ref. [75]. However, in principal one should to find out the expression for $\overset{o}{q}(x)$ solving the equation $q(x) = \overset{o}{q}(x) + \delta q[\overset{o}{q}(x)]$.

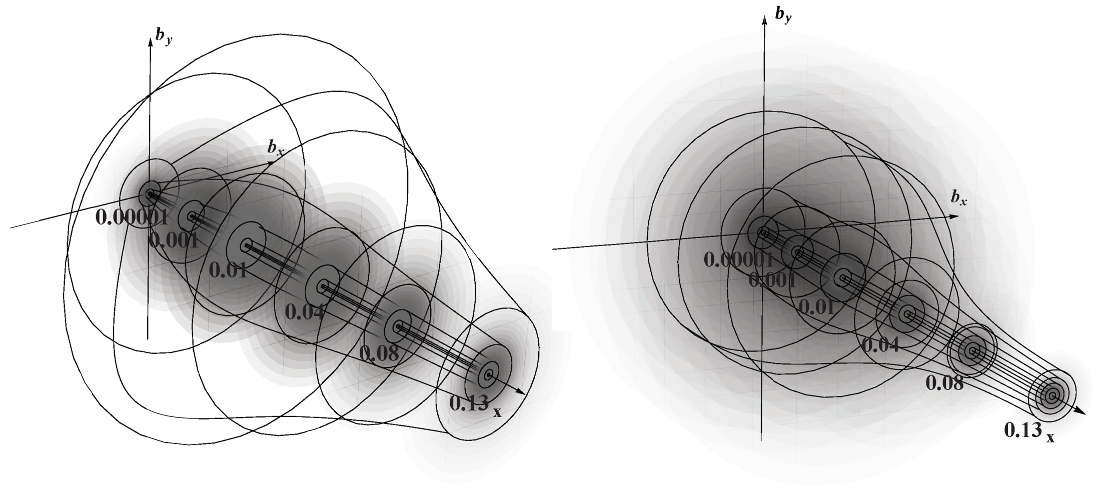

Figure IV-5: The density plot of $x\delta q(x,b)$ in within approximation (IV.2-87) for massive (left) and massless (right) cases.

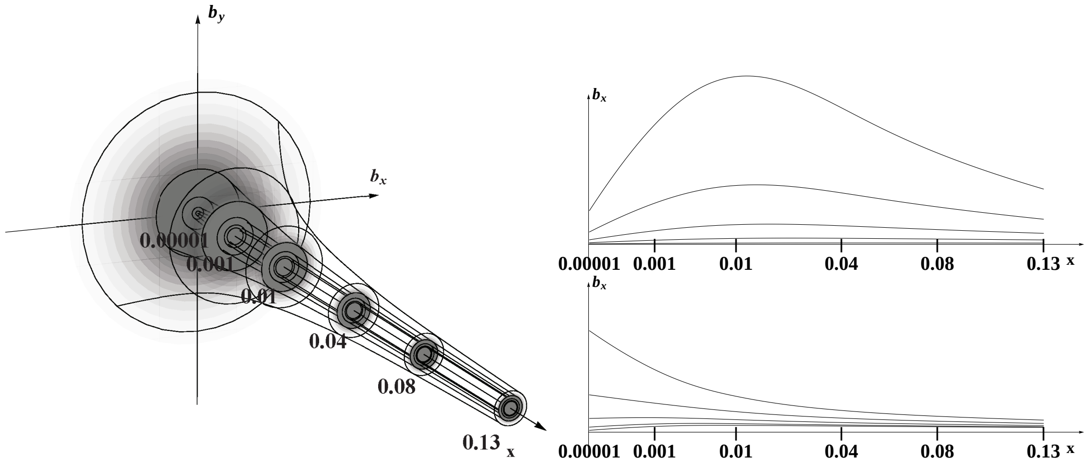

Figure IV-6: The density plot of $x\delta q(x,b)$ in within "Gribov diffusion" anzatz (left). The right plot is the comparison of isolines profiles of $x\delta q(x)$, right-up is the profile for fig.IV-5(left) density plot, right-down is the profile for "Gribov diffusion" ansatz.

diffusion" ansatz [65]:

$$q(x,\Delta^2) = q(x)e^{-\alpha'\Delta^2 \ln\left(\frac{1}{x}\right)}.$$

The comparison of our expression with the "Gribov diffusion" ansatz gives the following results: In the Gribov picture the pion impact distribution fall down as e^{-b^2}, in our calculations it falls down power-like (however, at large distances it is also exponentially suppressed, see (IV.2-79)); The most part of the pion radius in the Gribov picture is concentrated at $x \to 0$, in contrast to our result that is mainly concentrated at $x \sim \frac{m}{F}$. Therefore, shortly one can say that in the chiral expansion the pion is fatter at "not-so-small" $x \sim \frac{m}{F}$. The comparison of the "Gribov diffusion" ansatz and the chiral expansion are shown in fig.(IV-6). The average radiuses of pion for these two models are

$$\sqrt{\langle b^2 \rangle}_{\text{(IV.2-73)}} \simeq 0.3\text{fm}, \quad \sqrt{\langle b^2 \rangle}_{\text{Gribov}} \simeq 0.57\text{fm}.$$

Therefore, we can estimate the contribution of the higher chiral terms as 40%.

Conclusion

We have studied the logarithmical structure of the perturbative expansion in an EFT. It has been show that in massless EFTs the logarithms of the given order (i.e. leading order, next-to-leading order and so on) can be obtained at any order of the perturbative expansion, under foreseeable volume of calculations. Particulary, we have presented and investigated two independent methods for obtaining the LLog coefficients in a massles EFT.

The first method is based on the fact that in the massless EFT the coefficients of IR logarithms coincide with the coefficients of RG logarithms. The later can be found using the RG invariance (I.4-23), which for the non-renormalizable theories leads to the infinite set of the ordinary differential equations. Investigating the topological properties of graphs which give the RG coefficients, we have derived the general method to construct the recursive equations of the LLog coefficients. These equations present the generalization of the well-known RGEs for theories with an infinite number of interaction terms (non-renormalizable theories).

The structure of the recursive equations is directly connected with the topological structure of the one-loop graphs of the theory. Therefore, they have the same form for any theory of given topology and differ only by the kernels. The kernels are given by the LLog coefficient of the one-loop graphs with the vertices of arbitrary orders. Thus, one has to introduce the general form of the higher order Lagrangians. In the sector of 4-field interaction the higher order Lagrangian can be effectively presented through the Legendre polynomial (II.3-41) operator basis. This basis possess the "conformal-like" properties (A.1-7) which dramatically simplify the calculations.

We have presented in details the derivation of the recursive equations for the 4-point amplitude (II.3-36), for the FF (II.4-63) and for the 6-point amplitude (II.6-93), within an EFT with the lowest 4-field interaction. The kernels for these equation calculated within the Legendre basis are presented, e.g. (II.3-52,II.4-64,II.4-68). The calculation was done for models with different groups of isotopic symmetry: $O(N+1)/O(N)$, $SU(N)\times SU(N)$ and $O(N+K)/O(N)O(K)$.

The second method is based on the idea that the logarithms of the perturbative expansion cause the cuts in the complex plane of amplitude kinematic variables. Moreover, the two-particle cuts are related to LLogs, three-particle cuts are related to NLLog and so on. The properties of the cuts are fixed by the unitarity relation and by the crossing symmetry. That allows to find the relations between logarithm coefficients of different order of the perturbative expansion in the form of recursive equations (III.2-37), which is very similar to the RG recursive equation. Fixing of the tree-order of the amplitude, which enters to the recursive equation through the boundary condition, leads the fixing of LLogs at all orders.

The kernel of the recursive equation is composed from the crossing matrices both in group and momentum spaces (III.2-38). And it is general, in the sense that it is not depend on the particular realization of the Lagrangian, but only on the group of Lagrangian symmetries. The particular realization of EFT enters only through the boundary conditions.

We presented the expressions for wide classes of theories, including the renormalizable QFTs (III.2-39), QFTs in arbitrary even space-time dimension (III.2-47) and QFTs with the mixed interaction (III.2-41). Also we have explicitly demonstrated the method on the example of $O(N+1)/O(N)$, $SU(N)\times SU(N)$ σ-models in four and D dimensions.

The methods are independent of each other. Their relative advantages and disadvantages are followed:

- The method based on the RG invariance is more "fundamental", in the sense that its generalization on the other types of EFTs (such as the theories with the three-field interactions) is straightforward. In the contrast, the "analytical" method can not be applied for the EFTs with three-field interaction terms.

- The calculations of the equation kernel in the "analytical" method reduces to the pure algebraic operations, which can be done once for large classes of theories. On the other hand, the obtaining of the "RG" equation kernel supposes the independent loop-calculation for every QFT.

- The equation given by the "analytical" method (III.2-37) has the simpler structure, than the equation obtained by the "RG" method (II.3-36). The timing of the numerical evaluation of the recursive equation in the form (III.2-37) is several times lower than for the equation in the form (II.3-36). Also the simplicity of eqn. (III.2-37) gives up hope to solve it explicitly.

Both these methods produce the same results, that provides the strong cross-check of them. The results of calculations of the LLog coefficients in considered models are coincided with known 2-loop calculations [12],[42],[48], etc. (see citations in the text). Also the presented methods provide the correct large-N expansion of the considered theories. The obtained equations in the renormalizable theories give the well known results of the one-loop (or LLog) evolution.

The main problem of these methods is that they are not applicable to QFTs with masses. For the "RG" method, it is connected with the non-triviality of massive tadpole graphs. Tadpoles infinitely increase the number of graph topologies to be considered at given order of logarithms. This make the possibility to build close system of relations for logarithm coefficients impossible. For the "analytical" method the presence of mass means presence of the additional parameter in equations on coefficients. At the same time, the number of equations remains the same.

From the practical point of view the calculation of the LLog terms at high orders in an EFT is not important. The IR logarithmical rising weaken on the background of the power lowering. However, there is the large class of tasks where the logarithmical contributions are very important. These are the tasks of considering the chiral behavior of the non-local operators. And particulary there are light-cone operators, which play an important role in the hadron physics.

The cancelation of the power behavior by the light-cone distance at small-x region makes the LLog contribution much large than others. The presence of singularity, which is a consequence of the loop-convolution of local and non-local vertices, makes the task of all order summation necessary. Such summation of LLog singular terms was performed for pion partonic distributions with the help of methods mentioned above, (IV.2-44,IV.2-60). We have shown that such resummation restores the leading order chiral behavior of the distribution and gets rid of unphysical singularities.

The explicit resummation allowed us to reveal novel phenomena in the quark mass expansion of PDFs, low energy behavior of GPDs and amplitudes of hard exclusive processes. The main qualitative (model independent) results are the following:

- The leading small m_π asymptotic behavior in the region of small x of pion PDFs depends on the intercept (α) of the corresponding Regge trajectory:

  $$q(x) \sim \frac{1}{x^\alpha}\left[\frac{m_\pi^2}{(4\pi F_\pi)^2}\ln\left(\frac{1}{m_\pi^2}\right)\right]^\alpha .$$

  We have considered the Regge-like behavior of PDFs for simplicity, one can easily obtain corresponding leading chiral corrections for other types of small x behavior of PDFs;

- The leading small t behavior of the amplitude for hard exclusive processes on the pion target has the form:

  $$\mathrm{Im}\,\mathcal{A}(\xi,t) \sim \frac{1}{\xi^\alpha}\left[\frac{|t|}{4(4\pi F_\pi)^2}\ln\left(\frac{1}{|t|}\right)\right]^\alpha .$$

  Measurements of such processes at small x_{Bj} and small t would allow us to probe the chiral dynamics in a completely new regime – the dynamics of chiral and quark–gluon degrees of freedom intertwines.

- The leading large impact parameter ($b_\perp$) behavior of the quark distribution in the transverse plane can be obtained within the logarithmical accuracy (we show the result for $m_\pi = 0$, the result for $m_\pi \neq 0$ is given in eqn.(IV.2-79)):

$$q(x, b_\perp) \sim \frac{1}{x^\alpha} \frac{\ln^{\alpha-1}(b_\perp^2)}{(b_\perp^2)^{1+\alpha}} .$$

The distribution of quarks at large impact parameter is controlled completely by the all order resummed ChPT developed in the thesis. This asymptotic behavior is determined by the small-x behavior of usual PDFs, hence this asymptotic behavior depends on the scale, at which the corresponding PDF is defined. This is new and interesting result – the chiral expansion meets the QCD evolution.

- The transverse size of the pion is strongly increased at $x \sim \frac{m}{F}$. This effect (called "chiral inflation") is mainly responsible for the formation of the pion radius.

The complete results for the resummation of ChPT for pion PDFs and GPDs are given in the main body of the thesis.

There are many ways for a generalization and expansion of the presented methods. The most interesting of them are: the consideration of the EFT with fermions; the consideration of the non-renormalizable theories with three-field interaction terms; the considering of the QFT in odd dimension; the consideration of the next-to-leading effects and non-zero mass. The solution of these problems will allow one to clarify such important physical tasks as the calculation of the leading quantum correction to the gravitation processes, in particular Newton law, the calculation of the critical indices in the solids state physics and so on. The presented methods can be used as a simple and powerful tool for the investigation of the perturbative series.

A

Loop-calculation

A.1 The basic loop integral

The basic integral for the calculation of the one-loop β-function of $g^{(4)}$ coupling constant (II.3-52) has the following form:

$$A_{iA} * A_{n-i,B} = \int \frac{d^D l}{(2\pi)^D} \frac{A_{iA}(p_1, p_2, l-P, -l) A_{n-i,B}(l, P-l, p_3, p_4)}{l^2 (p_1 + p_2 - l)^2}, \quad \text{(A.1-1)}$$

where $D = 4 - 2\varepsilon$, $P = p_1 + p_2$, and the function A is given by exp. (II.3-42)

$$A_{nC} \;=\; i(-1)^{n+1} [(k_1 + k_2)^2]^n P_C \left(\frac{(k_1 - k_2) \cdot (k_3 - k_4)}{(k_1 + k_2)^2} \right).$$

For the calculation of the β-function we need to extract the pole coefficient of this integral.

Making the Wick rotation and general simplifications, one rewrites the integral (A.1-1) in the form

$$A_{iA} * A_{n-i,B} = i(-1)^B (-1)^{n+1} \left(P^2\right)^n \int \frac{d^D l}{(2\pi)^D} \frac{P_A\left(2\frac{(\Delta_1 l)}{P^2}\right) P_B\left(2\frac{(\Delta_2 l)}{P^2}\right)}{l^2 (P-l)^2}, \quad \text{(A.1-2)}$$

where we have introduced the notations

$$\Delta_1 = p_1 - p_2, \quad \Delta_2 = p_3 - p_4.$$

In exp. (A.1-2) we have neglected the terms proportional to $P\Delta_i$, because their counterterm is proportional to the $\partial^2\pi$-structure, see details in the text after exp. (II.3-39)

Using the explicit form of the Legendre polynomial

$$P_A(x) = \frac{(-1)^{\frac{3A}{2}}}{2^A} \sum_{n=0}^{A} \frac{(-1)^{\frac{n}{2}}(n+A)!x^n}{\left(\frac{A+n}{2}\right)!\left(\frac{A-n}{2}\right)!n!},$$

where the summation goes over even (odd) n for even (odd) A. We rewrite the basic integral as the double sum

$$\begin{aligned} A_{iA} * A_{n-i,B} &= i(-1)^{n+B+1}\frac{(P^2)^n}{(2\pi)^D}\frac{(-1)^{\frac{3}{2}(A+B)}}{2^{A+B}} \cdot \\ &\sum_{k,m=0}^{A,B} \frac{(k+A)!(m+B)!}{\left(\frac{A+k}{2}\right)!\left(\frac{A-k}{2}\right)!\left(\frac{B+m}{2}\right)!\left(\frac{B-m}{2}\right)!}\frac{I_{k,m}}{k!m!}, \end{aligned} \tag{A.1-3}$$

where

$$I_{n,m} = \frac{1}{(P^2)^{n+m}}\int d^Dl\frac{(\Delta_1 l)^n(\Delta_2 l)^m}{l^2(P-l)^2}.$$

This integral is considered in book [76] with the result:

$$\begin{aligned} I_{n,m} &= \frac{\pi^{D/2}}{(-P^2)^{\varepsilon}}\frac{\Delta_1^{\mu_1}..\Delta_1^{\mu_n}\Delta_2^{\nu_1}..\Delta_2^{\nu_m}}{(P^2)^{n+m}} \cdot \\ &\sum_{r=0}^{\frac{n+m}{2}} A_{NT}(1,1;r,n+m)\left(\frac{P^2}{2}\right)^r \{[g]^r[P]^{m+n-2r}\}^{\mu_1..\mu_n\nu_1..\nu_m}, \end{aligned} \tag{A.1-4}$$

where A_{NT} is the construction of Gamma-functions which will be given later, and $\{[g]^r[P]^{m+n-2r}\}^{\mu_1..\nu_m}$ is a totally symmetric tensor of $(m+n)$ rang composed from tensors $g_{\mu\nu}$ and vectors P_μ.

The contraction of the Δ_μ and P_μ does not contribute to the counterterm. Therefore, the terms with metric tensors contribute to the sum (A.1-4). The summation index r takes only the value $\frac{m+n}{2}$. The contractions of the Lorentz indices are given by the expression

$$\begin{aligned} &\Delta_1^{\mu_1}..\Delta_1^{\mu_n}\Delta_2^{\nu_1}..\Delta_2^{\nu_m}\{[g]^{\frac{n+m}{2}}\}^{\mu_1..\mu_n\nu_1..\nu_m} = \\ &\sum_{k=0}^{\min[m,n]} \frac{m!n!}{2^{\frac{m+n-2k}{2}}\left(\frac{n-k}{2}\right)!\left(\frac{m-k}{2}\right)!k!}\left(\Delta_1^2\right)^{\frac{n-k}{2}}\left(\Delta_2^2\right)^{\frac{m-k}{2}}\left(\Delta_1\Delta_2\right)^k, \end{aligned} \tag{A.1-5}$$

where the summation goes in such way that $(n-k)$ and $(m-k)$ are simultaneously even numbers.

The coefficient A_{NT} for $r = \frac{n+m}{2}$ is given by the expression

$$A_{NT}(1,1;\frac{n+m}{2},n+m) = \Gamma\left(\varepsilon - \frac{n+m}{2}\right) B\left(\frac{n+m}{2}+1-\varepsilon, \frac{n+m}{2}+1-\varepsilon\right)$$
$$= \frac{(-1)^{\frac{n+m}{2}}}{\varepsilon}\frac{\left(\frac{n+m}{2}\right)!}{(n+m+1)!} + \mathcal{O}(\varepsilon^0). \quad \text{(A.1-6)}$$

Substituting expressions (A.1-5) and (A.1-6) to exp. (A.1-3) and using that $\Delta_1^2 = \Delta_2^2 = -P^2$ one can perform the summation over k and m with the help of the hypergeometric series [77]. The result of the summation can be rewritten in the form of Legendre polynomial:

$$A_{iA} * A_{n-i,B} = \frac{i(-1)^A(-1)^{n+1}\left(P^2\right)^n}{(4\pi)^2\varepsilon} P_A\left(\frac{(\Delta_1\Delta_2)}{P^2}\right)\frac{\sin\left[\frac{A-B}{2}\pi\right]}{A(A+1)-B(B+1)} + \mathcal{O}(\varepsilon^0),$$

where A and B are supposed to be both even or odd, otherwise the sum gives zero. The right hand side of the expression is proportional to the Kronecker delta. Taking the limit $B \to A$ one obtains

$$A_{iA} * A_{n-i,B} = \frac{(-1)^A}{(2\pi)^2\varepsilon}\frac{\delta_{AB}}{2A+1}A_{nA} + \mathcal{O}(\varepsilon^0). \quad \text{(A.1-7)}$$

The loop-contraction of B and C structures (II.3-42), in the corresponded channel is also diagonal, i.e.

$$B_{iA} \overset{t}{*} B_{n-i,B} = \int \frac{d^Dl}{(2\pi)^D}\frac{B_{iA}(p_1,l,P-l,p_4)B_{n-i,B}(l-P,-l,p_2,p_3)}{l^2(p_1+p_4-l)^2} \quad \text{(A.1-8)}$$
$$= \frac{(-1)^A}{(2\pi)^2\varepsilon}\frac{\delta_{AB}B_{nA}}{2A+1} + \mathcal{O}(\varepsilon^0),$$

$$C_{iA} \overset{u}{*} C_{n-i,B} = \int \frac{d^Dl}{(2\pi)^D}\frac{C_{iA}(p_4,p_2,l,l-P)C_{n-i,B}(l-P,-l,p_3,p_1)}{l^2(p_2+p_4-l)^2} \quad \text{(A.1-9)}$$
$$= \frac{(-1)^A}{(2\pi)^2\varepsilon}\frac{\delta_{AB}C_{nA}}{2A+1} + \mathcal{O}(\varepsilon^0),$$

where the subscript above the asterisk denotes the loop integral corresponding to the second($\overset{t}{*}$) and the third ($\overset{u}{*}$) graphs in fig.2.3.

A.2 β-function in the $O(N+1)/O(N)$ σ-model

The one-loop $\beta^{(4)}$ function in the $O(N+1)/O(N)$ σ-model is given by the pole coefficients of the diagrams shown in fig. 2.3. The Feynman rule for the vertices was obtained in the chapter 2, (II.3-42). It reads

$$V^{abcd}_{nC}(k_1,k_2,k_3,k_4) \;=\; g_{nC}\left[\delta^{ab}\delta^{cd}A_{nC}+\delta^{ad}\delta^{bc}B_{nC}+\delta^{ac}\delta^{bd}C_{nC}\right], \qquad \text{(A.2-10)}$$

$$A_{nC} \;=\; i(-1)^{n+1}[(k_1+k_2)^2]^n P_C\left(\frac{(k_1-k_2)\cdot(k_3-k_4)}{(k_1+k_2)^2}\right),$$

$$B_{nC} \;=\; i(-1)^{n+1}[(k_1+k_4)^2]^n P_C\left(\frac{(k_1-k_4)\cdot(k_2-k_3)}{(k_1+k_4)^2}\right),$$

$$C_{nC} \;=\; i(-1)^{n+1}[(k_1+k_3)^2]^n P_C\left(\frac{(k_1-k_3)\cdot(k_2-k_4)}{(k_1+k_3)^2}\right).$$

The notation for the incoming momenta and indices are the same as shown in fig.2.4.

The s-channel diagram G_s (the left in fig.2.3.) gives the expression

$$\begin{aligned}&G_s(i,A;n-i,B) = \\ &\frac{g_{iA}g_{n-i,B}}{2}(\delta^{ab}\delta^{km}A_{iA}+\delta^{ak}\delta^{bm}B_{iA}+\delta^{am}\delta^{bk}C_{iA})*(\delta^{km}\delta^{cd}A_{n-i,B}+ \\ &\qquad\qquad +\delta^{kd}\delta^{mc}B_{n-i,B}+\delta^{kc}\delta^{md}C_{n-i,B}),\end{aligned}$$

where the asterisk denotes the loop-contraction in the s-channel, (A.1-1). The coefficient one-half instead of the expression is the symmetric coefficient for the diagram.

Contracting the group indices one obtains:

$$\begin{aligned}G_s \;=\; &\frac{g_{iA}g_{n-i,B}}{2}\Big[\delta^{ab}\delta^{cd}\cdot \qquad \text{(A.2-11)}\\ &(NA_{iA}*A_{n-i,A}+A_{iA}*B_{n-i,A}+A_{iA}*C_{n-i,A}+B_{iA}*A_{n-i,A}+C_{iA}*A_{n-i,A})\\ &\quad+\delta^{ad}\delta^{cd}\,(B_{iA}*B_{n-i,A}+C_{iA}*C_{n-i,A})\\ &\quad+\delta^{ac}\delta^{bd}\,(B_{iA}*C_{n-i,A}+C_{iA}*B_{n-i,A})\Big].\end{aligned}$$

In order to express exp. (A.2-11) through the basic integral (A.1-7), we introduce the expansion:

$$B_{nA} \;=\; \sum_{B=0}^{n}(-1)^A\Omega^{AB}_n A_{nB} \qquad \text{(A.2-12)}$$

$$C_{nA} \;=\; \sum_{B=0}^{n}(-1)^B\Omega^{AB}_n A_{nB} \qquad \text{(A.2-13)}$$

where the Ω_n^{AB} is given by exp. (II.3-48). Thus, exp. (A.2-11) can be rewritten as

$$G_s = \frac{g_{iA} g_{n-i,B}}{2} \sum_{J_1,J_2=0}^{i,n-i} A_{iJ_1} * A_{n-i,J_2} \Big[$$
$$\delta^{ab}\delta^{cd}\left(N\delta^{AJ_1}\delta^{BJ_2} + \left((-1)^{J_2}+(-1)^{B}\right)\delta^{AJ_1}\Omega_{n-i}^{BJ_2} + \left((-1)^{J_1}+(-1)^{A}\right)\Omega_i^{AJ_1}\delta^{BJ_2}\right)$$
$$+\delta^{ad}\delta^{bc}\left((-1)^{J_1+J_2}+(-1)^{A+B}\right)\Omega_i^{AJ_1}\Omega_{n-i}^{BJ_2}$$
$$+\delta^{ac}\delta^{bd}\left((-1)^{J_1+B}+(-1)^{A+J_2}\right)\Omega_i^{AJ_1}\Omega_{n-i}^{BJ_2}\Big].$$

Using basic loop-integral (A.1-7) one obtains

$$G_s = \frac{g_{iA} g_{n-i,B}}{(4\pi)^2\varepsilon} \sum_{J=0}^{n} \frac{A_{N,J}}{2J+1}\Bigg[\delta^{ab}\delta^{cd}\left(\frac{N}{2}\delta^{AJ}\delta^{BJ} + \delta^{AJ}\Omega_{n-i}^{BJ} + \Omega_i^{AJ}\delta^{BJ}\right)$$
$$+\delta^{ad}\delta^{cd}(-1)^J\Omega_i^{AJ}\Omega_{n-i}^{BJ} + \delta^{ac}\delta^{bd}\Omega_i^{AJ}\Omega_{n-i}^{BJ}\Bigg] + \mathcal{O}(\varepsilon^0),$$

where we have put $(-1)^A = (-1)^B = 1$, since A and B runs only through the even values.

Making the similar calculation for the last two diagrams in fig 2.4., and taking the sum of them one obtains the complete one-loop expresion

$$G = \frac{g_{iA} g_{n-i,B}}{(4\pi)^2\varepsilon} \sum_{J=0}^{n} \frac{\delta^{ab}\delta^{cd}}{2J+1}\Bigg[\tag{A.2-14}$$
$$\left(\frac{N}{2}\delta^{AJ}\delta^{BJ} + \delta^{AJ}\Omega_{n-i}^{BJ} + \Omega_i^{AJ}\delta^{BJ}\right)A_{nJ} + (-1)^J\Omega_i^{AJ}\Omega_{n-i}^{BJ}B_{nJ} + \Omega_i^{AJ}\Omega_{n-i}^{BJ}C_{nJ}\Bigg]$$
$$+\begin{pmatrix} b \leftrightarrow d \\ p_2 \leftrightarrow p_4 \end{pmatrix} + \begin{pmatrix} b \leftrightarrow c \\ p_2 \leftrightarrow p_3 \end{pmatrix} + \mathcal{O}(\varepsilon^0).$$

Expanding B_{nJ} and C_{nJ} in the square brackets over A_{nC} (A.2-12-A.2-13), one reproduces the expression for the vertex V_{nC} (A.2-10):

$$G(i,A;n-i,B) = \frac{g_{iA} g_{n-i,B}}{(4\pi)^2\varepsilon} \sum_{C=0}^{n} \beta(i,A;n-i,B/C)V_{nC} + \mathcal{O}(\varepsilon^0),$$

where the pole coefficient has the form

$$\beta(i,A;n-i,B/C) = \tag{A.2-15}$$
$$\frac{N}{2}\frac{\delta_{AB}\delta_{AC}}{2C+1} + \frac{\delta_{AC}\Omega_{n-i}^{BA} + \delta_{BC}\Omega_i^{AB}}{2C+1} + (1+(-1)^C)\sum_{J=0}^{\min[i,n-i]} \frac{\Omega_i^{AJ}\Omega_{n-i}^{BJ}\Omega_n^{JC}}{2J+1}.$$

It is the required one-loop expression for the $\beta^{(4)}$-function (II.2-14). The factor $(1+(-1)^C)$ makes the extraction of the odd C vertices in the Lagrangian.

We have presented the β functions of the other models (the $SU(N) \times SU(N)$ σ-model see the list after the formula (II.5-82), the matrix $O(N+K)/O(N) \times O(K)$ σ-model see the list after (II.5-85)) without an explicit derivation, since it differs only by the group indices convolutions. Also we note that in the matrix model the odd values of auxiliary indices are presented, and one should be careful with it during the calculation.

A.3 Calculation of anomalous dimensions

For the calculation of the anomalous dimensions for FF operators one needs to calculate the loop-integrals with open indices, particulary, with one and two open indices.

The usual algorithm of computation of such integrals is the projecting of the integral onto all possible Lorenz structures composed from vectors P and q. After the projecting the denominator of the resulted integral can be simplified using that $P\Delta_i \sim 0$ and that the terms $\sim l^2$ and $\sim (P-l)^2$ turns the loop-integral to zero. The only unusual term is (Δl). It can be absorbed to the Legandre polynomial using the Legandre recursive formula

$$xP_B(x) = \frac{B}{2B+1} P_{B+1}(x) + \frac{B}{2B+1} P_{B-1}(x).$$

After all one obtains the linear equation system with integrals in the form $A_{i,0} * A_{n-i,B}$, (A.1-7). The list of needed integrals is the following

$$\int \frac{l_\mu P_B\left(\frac{2l\Delta}{P^2}\right)}{l^2(P-l)^2} d^d l = \frac{P_\mu}{2\varepsilon}\delta_{B0} - \frac{\Delta_\mu}{6\varepsilon}\delta_{B1} \tag{A.3-16}$$

$$\begin{aligned} \int \frac{l_\mu l_\nu P_B\left(\frac{2l\Delta}{P^2}\right)}{l^2(P-l)^2} d^d l &= g_{\mu\nu}\frac{P^2}{\varepsilon}\Big[\frac{\delta_{2B}}{60} - \frac{\delta_{0B}}{12}\Big] \\ &+ \frac{P_\mu P_\nu}{\varepsilon}\Big[\frac{\delta_{B0}}{3} - \frac{\delta_{2B}}{60}\Big] + \frac{\Delta_\mu \Delta_\nu}{20\varepsilon}\delta_{B2} - \Big[\Delta_\mu P_\nu + P_\mu \Delta_\nu\Big]\frac{\delta_{B1}}{12\varepsilon} \end{aligned} \tag{A.3-17}$$

In this section we demonstrate explicitly the calculation of the anomalous dimension for the tensor operators (II.4-71)-(II.4-72). The anomalous dimension for the vector (II.4-67) and the scalar (II.4-60) operators was obtained in the same way.

There is only one diagram needed to be calculated, see fig.2.6. The loop integral for this diagram is of the s-channel type (A.1-1). Therefore, it is useful to expand the 4-pion vertex over $A_{n,C}$ (A.2-10) and multiply on δ^{ab} from the operator vertex:

$$\delta^{ab} V^{ab,cd}_{n-i,A} = \delta^{cd} \sum_{B=0}^{n-i} \left[N\delta^{AB} + \left((-1)^A + (-1)^B\right)\Omega^{AB}_{n-i}\right] A_{n-i,B}. \tag{A.3-18}$$

The loop-convolution of the currents $J^{\mu\nu}_{1,2}$ (II.4-71-II.4-72) can be obtained with exp.

(A.3-16-A.3-17)

$$J^{\mu\nu}_{1,i} * A_{n-i,B} = f_i g_{n-i,B} \frac{(-s)^n}{2\varepsilon} \Big[(g_{\mu\nu} q^2 - q_\mu q_\nu) \Big(\frac{\delta_{B2}}{30} - \frac{\delta_{B0}}{3} \Big) + P_\mu P_\nu \frac{\delta_{B2}}{10} \Big] + \mathcal{O}(\varepsilon^0), \tag{A.3-19}$$

$$J^{\mu\nu}_{2,i} * A_{n-i,A} = -h_i g_{n-i,B} \frac{(-s)^n}{2\varepsilon} \Big[(g_{\mu\nu} q^2 - q_\mu q_\nu) \Big(\frac{\delta_{B2}}{30} - \frac{\delta_{B0}}{6} \Big) + P_\mu P_\nu \frac{\delta_{B2}}{10} \Big] + \mathcal{O}(\varepsilon^0), \tag{A.3-20}$$

where we have used the notation for incoming momenta as in exp. (II.4-69), the factor one-half is the symmetry factor of diagram, fig.2.6.

The right hand sides of exp. (A.3-19-A.3-20) have to be expressed in the form of (II.4-71)-(II.4-72). The pole coefficients near the corresponding expressions give its anomalous dimension. Thus, the one-loop anomalous dimensions for the tensor operators are

$$\begin{aligned} Z^f &= f_i g_{n-i,A} Z^{ff} + h_i g_{n-i,A} Z^{fh} \qquad\qquad \text{(A.3-21)} \\ &= \sum_{B=0}^{n-i} \left[N\delta^{AB} + \left((-1)^A + (-1)^B \right) \Omega^{AB}_{n-i} \right] \\ &\qquad \left[f_i \Big(\frac{\delta_{B2}}{30} - \frac{\delta_{B0}}{3} \Big) + h_i \Big(\frac{\delta_{B0}}{6} - \frac{\delta_{B2}}{30} \Big) \right] g_{n-i,A}, \\ Z^h &= f_i g_{n-i,A} Z^{fh} + h_i g_{n-i,A} Z^{hh} \qquad\qquad \text{(A.3-22)} \\ &= \sum_{B=0}^{n-i} \left[N\delta^{AB} + \left((-1)^A + (-1)^B \right) \Omega^{AB}_{n-i} \right] \\ &\qquad \left[f_i \Big(\frac{\delta_{B2}}{15} - \frac{\delta_{B0}}{3} \Big) + h_i \Big(\frac{\delta_{B0}}{6} - \frac{\delta_{B2}}{15} \Big) \right] g_{n-i,A}. \end{aligned}$$

In the matrix form these expressions are

$$\begin{pmatrix} Z^{ff} & Z^{fh} \\ Z^{hf} & Z^{hh} \end{pmatrix} = \left(N\delta^{AB} + 2\Omega^{AB}_{n-i} \right) \left[\frac{\delta_{B0}}{3} \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} + \frac{\delta_{B2}}{15} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix} \right]. \tag{A.3-23}$$

The anomalous dimensions for the scalar and the vector currents are presented in (II.4-64) and (II.4-68) respectively.

B

Matrix Ω

B.1 Properties of Ω in $D = 4$

The operator $\hat{\Omega}$ implements the crossing transformation in the momentum space of the 4-particle amplitude between s- and t-channels:

$$\hat{\Omega}A(s,t,u) = A(t,s,u), \tag{B.1-1}$$

where s, t and u are usual Mandelshtam variables, connected by the momentum conservation law: $s+t+u=0$.

We are interested in the LLog realization of the operator $\hat{\Omega}$. Toward this aim we introduce the matrix Ω_n in the space of partial waves, which realizes the crossing transformation of the n'th order of the momentum expansion , i.e.

$$\begin{aligned} A(t,s,u) &= \sum_{n=0}^{\infty}\sum_{A=0}^{n} a_{nA} t^n P_A\left(1+\frac{2s}{t}\right) \\ &= \sum_{n=0}^{\infty}\sum_{A,B=0}^{n} a_{nA}\Omega_n^{AB} s^n P_B\left(1+\frac{2t}{s}\right) = \hat{\Omega}A(s,t,u). \end{aligned} \tag{B.1-2}$$

Therefore, the matrix element Ω_n^{AB} satisfies the relation

$$\frac{(\eta-1)^n}{2^n} P_A\left(\frac{\eta+3}{\eta-1}\right) = \sum_{B=0}^{n} \Omega_n^{AB} P_B(\eta),$$

where η is the cosine of the s-channel scattering angle in the c.m.s. system, $\eta = 1+\frac{2t}{s}$. Using the orthogonality of the Legandre basis one obtains the integral representation for matrix element of Ω_n

$$\Omega_n^{AB} = \frac{(2B+1)}{2^{n+1}} \int_{-1}^{1} dx\ P_A\left(\frac{x+3}{x-1}\right)(x-1)^n P_B(x). \qquad \text{(B.1-3)}$$

The indices A and B are constrained from 0 to n. Thus, Ω_n is a $(n+1)\times(n+1)$ matrix.

The operator $\hat{U}$ implements the $(t \leftrightarrow u)$ crossing transformation. Its matrix realization is the following

$$\hat{U}A(s,t,u) = A(s,u,t), \quad U^{AB} = \delta^{AB}(-1)^A. \qquad \text{(B.1-4)}$$

The definitions of crossing transformations (B.1-1) and (B.1-4) imply the following relations between the operators

$$\hat{\Omega}\hat{\Omega} = \hat{U}\hat{U} = I, \quad \hat{\Omega}\hat{U}\hat{\Omega} = \hat{U}\hat{\Omega}\hat{U},$$

where I is the equivalence transformation. In terms of matrices these relations read

$$\sum_{J=0}^{n} \Omega_n^{AJ}\Omega_n^{JB} = \delta^{AB}, \quad \sum_{J=0}^{n} \Omega_n^{AJ}(-1)^J\Omega_n^{JB} = (-1)^{A+B}\Omega_n^{AB}. \qquad \text{(B.1-5)}$$

These expressions can be checked by the direct computation of sums using the integral representation (B.1-3) and the completeness relation of the Legendre polynomials.

For numerical computations one needs the explicit expression for the Ω_n matrix element, which can be presented through the hypergeometric function of the Saalschutz form (see e.g.[77])

$$\Omega_n^{AB} = \frac{(-1)^{B+n}(2B+1)}{(n+B+1)!}\frac{n!n!}{(n-B)!}\ {}_4F_3\left(\begin{matrix} -A, A+1, -B-n-1, B-n \\ -n, -n, 1\end{matrix}\ \middle|\ 1\right). \qquad \text{(B.1-6)}$$

The Saalschutz form of the hypergeometric function allows one to obtain particular values of Ω through the elementary functions, e.g.

$$\Omega_n^{0B} = (-1)^{n+B}\frac{(2B+1)\ n!n!}{(n+B+1)!(n-B)!}, \quad \Omega_n^{A0} = (-1)^{n+A}\frac{(n-A)!(k+A+1)!}{(n+1)!(n+1)!},$$
$$\Omega_n^{An} = \frac{n!n!}{(2n)!}, \quad \Omega_n^{nB} = (2B+1)(2n+1)!\left[\frac{n!}{(n-B)!(n+B+1)!}\right]^2.$$

The following relations often appear during the consideration of recursive equations

(III.2-37), (II.4-63)

$$\sum_{J=0}^{n}\Omega_n^{BJ}=\delta_{n,A}\frac{(2n)!}{n!n!}\ ,\qquad \sum_{J=0}^{n}(-1)^J\Omega_n^{BJ}=(-1)^{n+B} \tag{B.1-7}$$
$$\sum_{J=0}^{\min[i,n-i]}\frac{\Omega_i^{0J}\Omega_{n-i}^{0J}\Omega_n^{J0}}{2J+1}=\frac{(n-i)!(n-i)!i!i!}{(n+1)!(n+1)!}=B^2(i,n-i)$$
$$\sum_{C=0}^{n}\frac{\Omega_n^{BC}}{2C+1}=\frac{(-1)^{n+B}}{2}\frac{\Gamma(n+\frac{1}{2}-B)\Gamma(n+\frac{3}{2}+B)}{\Gamma^2(n+\frac{3}{2})}$$

Often one needs to transform the partial expansion of amplitude (B.1-2) to the power expansion:

$$A(s,t,u)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}a_{nl}s^nP_l\left(1+\frac{2t}{s}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\chi_{nk}s^{n-k}t^k. \tag{B.1-8}$$

The coefficients χ_{nk} are connected with the coefficients a_{nl} in the following way

$$\chi_{nk}=\sum_{l=k}^{n}\frac{1}{k!k!}\frac{(k+l)!}{(l-k)!}a_{nl} \tag{B.1-9}$$
$$a_{nl}=\sum_{k=0}^{n}\Omega_k^{0,l}\chi_{nk},$$

The crossing relations for power decompositions leads to the following useful relations

$$\sum_{k=C}^{J}\Omega_k^{0C}\frac{(J+k)!}{(J-k)!}\frac{1}{k!k!}=\delta_{JC}\ ,\qquad \sum_{C=k}^{l}\Omega_l^{0C}\frac{(C+k)!}{(C-k)!}\frac{1}{k!k!}=\delta_{kl},$$

$$\sum_{k=0}^{J}\Omega_{n-k}^{0C}\frac{(J+k)!}{(J-k)!}\frac{1}{k!k!}=\Omega_n^{JC}\ ,\qquad \sum_{k=0}^{n-C}\Omega_{n-k}^{0C}\frac{(J+k)!}{(J-k)!}\frac{1}{k!k!}=\Omega_n^{JC},$$

$$\sum_{C=0}^{n}\Omega_{n-k}^{0C}\Omega_n^{CA}=\Omega_k^{0A}\ ,\qquad \sum_{C=0}^{n}\Omega_{n-k}^{AC}\Omega_n^{C0}=\frac{\Omega_k^{0A}}{2A+1}. \tag{B.1-10}$$

B.2 Properties of $\Omega_{n,D}$

In the D dimensions the partial wave expansion is defined through the Gegenbauer polynomials (III.2-46). The realization of the crossing operator (B.1-1) can be done in the similar way:

$$\begin{aligned} A(t,s,u) &= \sum_{n=0}^{\infty}\sum_{A=0}^{n} a^D_{nA} t^n C_A^{\frac{D-3}{2}}\left(1+\frac{2s}{t}\right) \\ &= \sum_{n=0}^{\infty}\sum_{A,B=0}^{n} a^D_{nA}\Omega^{AB}_{n,D} s^n C_B^{\frac{D-3}{2}}\left(1+\frac{2t}{s}\right) = \hat{\Omega}A(s,t,u). \quad \text{(B.2-11)} \end{aligned}$$

Therefore, the matrix element of $\Omega_{n,D}$ satisfies

$$\left(\frac{z-1}{2}\right)^n C_A^{\frac{D-3}{2}}\left(\frac{z+3}{z-1}\right) = \sum_{B=0}^{n}\Omega^{AB}_{n,D} C_B^{\frac{D-3}{2}}(z). \quad \text{(B.2-12)}$$

The orthogonality of the Gegenbauer polynomials leads to the integral representation for the matrix element

$$\begin{aligned} \Omega^{AB}_{n,D} &= \frac{2B+D-3}{2}\frac{2^{D-4}B!}{\Gamma(B+D-3)} \quad \text{(B.2-13)} \\ & \frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi}\cdot\int_{-1}^{1}(1-z^2)^{\frac{D-4}{2}}\left(\frac{z-1}{2}\right)^n C_A^{\frac{D-3}{2}}\left(\frac{z+3}{z-1}\right) C_B^{\frac{D-3}{2}}(z)dz. \end{aligned}$$

The indices A and B runs from 0 to n.

The matrix U, which realize the $(t \leftrightarrow u)$ crossing transformation, is the same as for $D=4$ case, (B.1-4).

The group properties of the crossing transformations lead to the relations, similar to (B.1-5),

$$\sum_{B=0}^{n}\Omega^{AB}_{n,D}\Omega^{BC}_{n,D} = \delta^{AC}, \quad \sum_{B=0}^{n}\Omega^{AB}_{n,D}(-1)^B\Omega^{BC}_{n,D} = (-1)^{A+C}\Omega^{AC}_{n,D}. \quad \text{(B.2-14)}$$

For numerical computations it is convenient to represent the $\Omega_{n,D}$ matrix through the hypergeometric function of the Saalschutz form:

$$\begin{aligned} \Omega^{AB}_{n,D} &= \frac{(-1)^{B+n}(2B+D-3)}{\Gamma(n+B+D-2)}\frac{n!}{(n-B)!}\frac{\Gamma(A+D-3)}{A!}\frac{\Gamma(n+\frac{D-2}{2})}{\Gamma(\frac{D-2}{2})} \quad \text{(B.2-15)} \\ & {}_4F_3\left(\left.\begin{matrix} -A, A+D-3, -B-n-D+3, B-n \\ -n, -n-\frac{D-4}{2}, \frac{D-2}{2} \end{matrix}\right| 1\right), \end{aligned}$$

The often meeting particular values of $\Omega_{n,D}$ are

$$\begin{aligned} \Omega^{0B}_{n,D} &= (-1)^{n+B}2^{D-4}(2B+D-3)\frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}}\frac{n!\Gamma\left(n+\frac{D-2}{2}\right)}{(n-B)!\Gamma(n+B+D-2)}, \\ \Omega^{A0}_{n,D} &= (-1)^{n+A}\frac{(D-3)\Gamma(A+D-3)}{\Gamma(n+D-2)A!}\frac{\Gamma\left(n+A+\frac{3D-8}{2}\right)\Gamma\left(n-A+\frac{D-2}{2}\right)}{\Gamma\left(n+\frac{3D-8}{2}\right)\Gamma\left(\frac{D-2}{2}\right)}, \end{aligned}$$

$$
\begin{aligned}
\Omega^{An}_{n,D} &= \frac{n!\Gamma\left(n+\frac{D-2}{2}\right)}{\Gamma\left(2n+D-3\right)\Gamma\left(\frac{D-2}{2}\right)}\frac{\Gamma(A+D-3)}{A!},\\
\Omega^{nB}_{n,D} &= \frac{(2B+D-3)\Gamma(n+D-3)}{(n-B)!\Gamma(n+B+D-2)}\frac{\Gamma\left(n+\frac{D-2}{2}\right)\Gamma\left(2n+\frac{3D-8}{2}\right)}{\Gamma\left(n-B+\frac{D-2}{2}\right)\Gamma\left(n+B+\frac{3D-8}{2}\right)}.
\end{aligned}
$$

At the large D the matrix $\Omega_{n,D}$ has an asymptotic behavior as $\Omega^{AB}_{n,D} \sim D^{A-B}$.

The relation between the power expansion in D dimensions and the partial wave decomposition is given by

$$
\begin{aligned}
\chi_{nk} &= \sum_{l=k}^{\frac{D-4}{2}(n-1)} \frac{4^l\Gamma\left(\frac{D-3}{2}+k\right)\Gamma\left(l+k+D-3\right)}{k!\Gamma\left(\frac{D-3}{2}\right)\Gamma(2k+D-3)(l-k)!}a_{nl}\\
a_{nl} &= \sum_{k=0}^{\frac{D-4}{2}(n-1)} \Omega^{0,l}_{k,D}\chi_{nk},
\end{aligned}
$$

where χ_{nk} is the coefficient near the $t^k s^{\frac{(D-4)}{2}(n-1)-k}$ structure.

C

Properties of recursive equation

C.1 Solution in form of large-N expansion for $O(N)$ models

The models with $O(N)$-symmetry are solvable in the large-N limit. On diagrammatic level the leading large-N contribution corresponds to accounting only the chain diagrams, fig.2.4. The next-to-leading orders of large-N expansion are given by the addition of the chains to diagrams. At that the number of chains in a diagram corresponds to the order of large-N expansion. The calculation of such diagrams is a difficult task in EFT, because the divergenceness of any loop-integral makes the R-operation over diagrams with many loops highly non-trivial, even at the LLog approximation. However, with the help of recursive equation (II.3-36) the expression for the LLog part of large-N expansion can be found explicitly order by order without difficult calculations.

The LLog coefficient ω_{nC} is given by eqn. (II.3-36). The initial value for the iteration is $\omega_{10} = 1$. The β-function for the model was calculated in the second chapter (II.3-52) and has the form

$$\beta(i, A; n-i, B/C) \;=\; \frac{N}{2}\beta_0 + \beta_1 \tag{C.1-1}$$

where

$$\begin{aligned}\beta_0 &= \frac{\delta_{AB}\delta_{AC}}{2C+1},\\ \beta_1 &= \frac{\delta_{AC}\Omega^{BA}_{n-i} + \delta_{BC}\Omega^{AB}_{i}}{2C+1} + 2\frac{1+(-1)^C}{2}\sum_{J=0}^{\min[i,n-i]}\frac{\Omega^{AJ}_{i}\Omega^{BJ}_{n-i}\Omega^{JC}_{n}}{2J+1}.\end{aligned}$$

The solution of eqn. (II.3-36) is a polynomial over $\frac{N}{2}$, with the maximal power $(n-1)$, i.e.

$$\omega_{nC} = \left(\frac{N}{2}\right)^{n-1} \omega_{nC}^{(0)} + \left(\frac{N}{2}\right)^{n-2} \omega_{nC}^{(1)} + ... + \omega_{nC}^{(n-1)}. \qquad \text{(C.1-2)}$$

Substituting this ansatz to eqn. (II.3-36), and collecting the terms near the equal powers of $N/2$ one obtains a set of recursive equations

$$\begin{aligned}
\left(\frac{N}{2}\right)^{n-1} &: \omega_{nC}^{(0)} = \frac{1}{n-1}\sum_{i,A,B} \beta_0 \omega_{iA}^{(0)} \omega_{n-i,B}^{(0)} , \qquad \text{(C.1-3)}\\
\left(\frac{N}{2}\right)^{n-2} &: \omega_{nC}^{(1)} = \frac{1}{n-1}\sum_{i,A,B} \beta_0 \left(\omega_{iA}^{(1)} \omega_{n-i,B}^{(0)} + \omega_{iA}^{(0)} \omega_{n-i,B}^{(1)}\right) + \frac{1}{n-1}\sum_{i,A,B} \beta_1 \omega_{iA}^{(0)} \omega_{n-i,B}^{(0)} ,\\
&\vdots \qquad \vdots \qquad \vdots\\
\left(\frac{N}{2}\right)^{n-k-1} &: \omega_{nC}^{(k)} = \frac{1}{n-1}\sum_{i,A,B} \beta_0 \sum_{j=0}^{k} \omega_{iA}^{(j)} \omega_{n-i,B}^{(k-j)} + \frac{1}{n-1}\sum_{i,A,B} \beta_1 \sum_{j=0}^{k-1} \omega_{iA}^{(j)} \omega_{n-i,B}^{(k-j-1)} ,\\
&\vdots \qquad \vdots \qquad \vdots ,
\end{aligned}$$

where $\sum_{i,A,B} = \sum_{i=1}^{n-1}\sum_{A=0}^{i}\sum_{B=0}^{n-i}$. The boundary conditions are $\omega_{n,C}^{k\geqslant n} = 0$ and $\omega_{10}^{(0)} = 1$. All these equations, except the first one, are linear. Therefore, they can be solved by regular methods.

The main problem in the large-N expansion produces only the leading term. However, for the $O(N)$-type models, the solution for the leading term can be easily found. The equation for $\omega^{(0)}$ has the form of the RG equation of a renormalizable theory, because its kernel does not depend on n. Thus, one can convince that the solution for the leading large-N terms is

$$\omega_{nC}^{(0)} = \delta_{C0}. \qquad \text{(C.1-4)}$$

This result is well-known and can be found in many textbooks, e.g. [32]

Such simple leading-N solution makes a big simplification on all other orders of expansion. In the off-leading equations in the system (C.1-3) the unknown ω is contracted with $\omega_{nC}^{(0)}$ and β_0 only. Therefore, the equation for $\omega^{(k)}$ has the form

$$\omega_{nC}^{(k)} = \frac{2\delta_{C0}}{n-1}\sum_{i=1}^{n-1} \omega_{i0}^{(k)} + f_C^{(k)}(n), \qquad \text{(C.1-5)}$$

where

$$f_C^{(k)}(n) \;=\; \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{k-1} \frac{\omega_{iC}^{(j)}\omega_{n-i,C}^{(k-j)}}{2C+1} + \frac{1}{n-1}\sum_{i,A,B} \beta_1 \sum_{j=0}^{k-1} \omega_{iA}^{(j)} \omega_{n-i,B}^{(k-j-1)}.$$

The procedure of solution for this equation is discussed in the next section. The result is

$$\omega_{nC}^{(k)} \ = \ f_C^{(k)}(n) + \delta_{C0} \sum_{m=1}^{n-1} \frac{2n}{(m+1)m} f_0^{(k)}(m). \tag{C.1-6}$$

This expression allows one to find by recursion the needed order of large-N expansion. The expressions agree with the direct NLO large-N calculations.

In particular, we need the $\omega_{nn}^{(1)}$ value for the GPD consideration. It reads

$$w_{nn}^{(1)} \ = \ \frac{2}{n+1} \frac{n!n!}{(2n)!} \ , \quad n = \text{odd}. \tag{C.1-7}$$

C.2 On the solution of recursive equations

The recursive equations often appeared in the present thesis. All equations presented in the thesis have unique solutions. But there is no standard receipt for their solving. In the present section we show some methods for the investigation of the recursive equations. The mathematical literature is not voluminous, and it mainly concentrates on the linear equations, e.g. [52], [78].

First of all, let us consider the linear recursive equations. These equations are describes many "daughter" quantities, e.g. the equations for LLog coefficients for FF (II.4-63), for 6π-amplitude (II.6-93), for the off-leading orders of large-N expansion (C.1-5). We consider only the general structure of the equations without concretizing the kernels.

Let us consider a non-homogenous linear equation of the form

$$v_n = \frac{1}{n} \sum_{i=0}^{n-1} Z_{n-i} v_i + B_n, \tag{C.2-8}$$

where Z_{n-i} and B_n are known functions, $v_0 = 1$. This is a typical form of equation which appears in applications.

The solution of eqn. (C.2-8) is unique, since it is the infinite system of linear equations with a triangle matrix. It can be transformed to the integral Volterra equation of the second kind on the generating function

$$\Upsilon(x) - \Upsilon(0) \ = \ \int_0^x \Upsilon(y) W(y) \ dy + B(x) \ , \tag{C.2-9}$$

where

$$\Upsilon(x) = \sum_{n=0}^{\infty} v_n x^n \ \ , \ W(x) = \sum_{n=1}^{\infty} x^{n-1} Z_n \ , \ \ B(x) = \sum_{n=0}^{\infty} B_n x^n .$$

The boundary conditions is $\Upsilon(0) = v_0$. The general solution of eqn. (C.2-9) is following

$$\Upsilon(x) = \exp\left[\int_0^x W(y)\, dy\right]\left[\Upsilon(0) + \int_0^x B'(y)\exp\left(-\int_0^y W(z)\, dz\right)dy\right] \quad \text{(C.2-10)}$$

where $B' = \frac{d}{dx}B(x)$.

In the particular case, when the kernel is a number, $Z_n = a$, the expression (C.2-10) gives the generating function

$$\Upsilon(x) = \frac{1}{(1-x)^a}\left[\Upsilon(0) + \int_0^x B'(y)(1-y)^a dy\right].$$

Thus, the expression for v_n is

$$v_n = v_0\frac{(a)_n}{n!} + B_n + \sum_{k=1}^{n-1} B_k \frac{\Gamma(n-k+a)}{(n-k)!\Gamma(a)}\, {}_3F_2\Big(\{-a, k, k-n\}, \{k+1, k-n-a+1\}, 1\Big).$$

For values $a = 1, 2$ it reads

$$a = 1 \; : \; v_n = v_0 + B_n + \sum_{k=1}^{n-1}\frac{B_k}{k+1},$$
$$a = 2 \; : \; v_n = (n+1)v_0 + B_n + \sum_{k=1}^{n-1} B_k\frac{2(n+1)}{k^2+3k+2}.$$

Usually, one does not need to obtain the expression for v_n explicitly, because the generation function is more useful for the physical application.

Also the following recursive equation is interesting

$$v_n = \frac{\lambda_n}{n}\sum_{i=0}^{n-1} Z_{n-i}v_i + B_n,$$

the corresponded integral equation is

$$\Upsilon(x) - \Upsilon(0) = \int_0^x \Upsilon(y)\mathcal{W}(x,y)dy + B(x)$$

where

$$\Upsilon(x) = \sum_{n=0}^{\infty} v_n x^n \; , \quad \mathcal{W}(x,y) = W(y)\int_y^x \frac{dz}{z}\lambda\left(\frac{z}{y}\right) \; , \quad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

$$W(x) = \sum_{n=1}^{\infty} x^{n-1}Z_n \; , \quad \int_0^1 x^{n-1}\lambda(x) = \lambda_n.$$

This equation has no general solution for an arbitrary kernel, hence it has to be considered for every case separately.

In the cases when the auxiliary indices are presented the addition variable in the generating function has to be introduced. The example of such calculation can be found in the text, (III.1-14).

The recursive equations with quadratic non-linearity, e.g.(II.3-36), also can be expressed as an integral equation for generating functions. Schematically the usual recursive equation has the form

$$\omega_n = \frac{\lambda_n}{n-1}\sum_{i=1}^{n-1}\omega_i\omega_{n-i}. \tag{C.2-11}$$

The integral analog of this equation is a Volterra equation of the second kind in form of Hammerstein:

$$f(x) - f(0) = \int_0^x g\left(\frac{y}{x}\right) f^2(y)dy, \tag{C.2-12}$$

where

$$f(x) \; = \; \sum_{n=1}^{\infty} x^{n-1}\omega_n \;, \quad f(0) = \omega_1 \;, \quad \lambda_n = \int_0^1 g(y)y^{n-2}dy.$$

The Hammerstain equation has an unique solution. In some particular cases this equation is exactly solvable [53].

Since we can not present the general solution of eqn. (C.2-12), let us discuss the asymptotic properties of the solution. We do not have an unambiguous result for the asymptotic behavior. The non-linearity of the equation provides many uncontrollable effects. However, we present here some attempts to investigate the asymptotic behavior of the generating function and coefficients ω_n.

Let us suppose that the β-function has a weak dependence on n. Thus, one can approximate it as $\lambda_n = \beta + \frac{\delta}{2}n$. In this case the solution can be found in terms of the inverse function, and it has the form

$$x = \frac{f^{\frac{\beta}{\beta+\delta}} - 1}{\beta f},$$

which gives

$$\begin{aligned} f(x) \; &= \; \frac{1}{1-\beta x} - \frac{\delta}{\beta}\frac{\ln(1-\beta x)}{(1-\beta x)^2} \\ &+ \frac{\delta^2}{2\beta^2}\frac{\ln(1-\beta x)}{(1-\beta x)^3}\big((1+\beta x)\ln(1-\beta x) - 2x\beta\big) + \mathcal{O}\left(\left(\frac{\delta}{\beta}\right)^3\right). \end{aligned} \tag{C.2-13}$$

The first term represents the usual renormalizable solution. All others give the non-renormalizable corrections.

From this example we can conclude that the solution for the whole equation can be written in a form

$$f(x) \sim \frac{1}{(1-\beta x)}\left(1+\frac{1}{(1-\beta x)^{1+\epsilon}}+..\right), \quad \text{for } \lambda_n-\lambda_{n-1}\sim\epsilon\ll 1.$$

This estimation is also supported by the following consideration. Instead of an integral equation one can write a high order differential equation for the generating function $f(x)$. Expanding λ_n in Tailor series at the point $n=1$, $\lambda_n=\sum_k (n-1)^k b_k/k!$, we obtain the equation

$$xf' = \sum_{k=0}^{\infty} b_k (x\partial_x)^k \left[xf^2\right]. \tag{C.2-14}$$

This is a homogenous equation with the power of homogeneity equal (-1). Thus, the expected solution should have a form of $f\sim(1-\beta x)^{-1}$ and some addition "dimensionless" x-dependence. In the case then $b_k=\epsilon^k\tilde{b}_k$, where $\epsilon\ll 1$, the solution can be presented as an expansion in ϵ,

$$f=\frac{1}{1-b_0x}-\varepsilon\frac{\tilde{b}_1 b_0 x+2\ln(1-b_0x)}{b_0(1-b_0x)^2}+\mathcal{O}(\epsilon)\ .$$

At large x the leading terms in such expansion behave as $c_n\frac{x^{n-1}}{(1-b_0x)^n}$. The coefficients c_n can be found explicitly. Skipping the details we obtain

$$f(x)=\frac{1}{1-b_0x}\left(1+\sum_{k=1}^{\infty}\frac{b_k}{b_0}\left(\frac{b_0x}{1-b_0x}\right)^k\right)^{-1}+\mathcal{O}(x^{-2+\epsilon})=\frac{1}{1-\lambda_1 x}\frac{\lambda_1}{\lambda_{\frac{1}{1-b_0x}}}+\mathcal{O}(x^{-2+\epsilon}).$$

This expression gives an asymptotic behavior of the solution at large x for non-growing λ_n.

The more interested object of investigation is the asymptotic behavior of ω_n at large-n. It is given by the nearest to origin singularity of the generating function, e.g. see [79]. Since eqn. (C.2-14) is homogenous, the singularity of solution can be only a simple pole. Therefore, the asymptotic behavior is just

$$\omega_n \sim a^{n-1}, \tag{C.2-15}$$

where a is an inverse distance from zero to the pole. This number can be found using the Kovalevsky method. But the movement of the pole positions is very difficult for any more or less real β-function. Therefore, the task to find value a is still unsolved.

The value of a can be found when the n-dependence of λ_n is weak. In this case $a=\lambda_1$. We have already seen this result in the large-N expansion and in eqn. (C.2-13). This estimation also agrees with numerical calculations.

In the opposite case, when the λ_n is a fast increasing function of n, like $\lambda_n \gtrsim n\Delta^n$ and $\Delta \gg 1$, the leading behavior of ω_n is

$$\omega_n = \frac{2^{n-1}}{(n-1)!} \prod_{i=2}^{n} \lambda_i \left(1 + \mathcal{O}(\Delta^{-1})\right) .$$

We see that such behavior is of an absolutely another type than (C.2-15).

D

The unitarity relation in D-dimension

In this appendix we give a summary of definitions and relations for the partial wave decomposition, in the arbitrary even space-time dimension D.

The S matrix element in D dimensions is defined as:

$$\begin{aligned}\langle \phi^d(p_4)\phi^c(p_3)|S|\phi^b(p_2)\phi^a(p_1)\rangle &= \\ &= I^{abcd}_{p_1p_2p_3p_4} + i(2\pi)^D\delta^D(p_1+p_2+p_3+p_4)\sum_I P_I^{abcd}T_I(s,t,u),\end{aligned} \tag{D.1-1}$$

where P_I^{abcd} is the projector on the invariant subspace of the particle symmetry group (see a proper definition after (III.2-19)), T^I is the scattering amplitude. The first term corresponds to the identity part of the S matrix:

$$I^{abcd}_{p_1p_2p_3p_4} \;=\; (2\pi)^{2D-2}\,2p_{10}\,2p_{20}\,\delta^{ac}\delta^{bd}\delta^{D-1}(\vec{p}_1-\vec{p}_3)\delta^{D-1}(\vec{p}_2-\vec{p}_4),$$

where we have assumed that particles ϕ are commuting, i.e. $|\phi^a(p_1)\phi^b(p_2)\rangle = |\phi^b(p_2)\phi^a(p_1)\rangle$.

The unity in the space of fields ϕ has the form

$$I = \sum_{n=1}^{\infty}\prod_{i=1}^{n}\sum_{a_i}\int \frac{d^{D-1}p_i}{2p_{i0}(2\pi)^{D-1}}|\phi^{a_n}(p_n)..\phi^{a_1}(p_1)\rangle\langle\phi^{a_1}(p_1)..\phi^{a_n}(p_n)|.$$

Omitting the multi-particle states, and using the unitarity of S matrix, $SS^+ = I$, one obtains the elastic unitarity relation in the momentum space

$$
\begin{aligned}
&\mathrm{Im}\sum_I P_I^{abcd} T^I(p_1,p_2,p_3,p_4) \qquad \text{(D.1-2)}\\
&= \frac{1}{2(2\pi)^{D-2}} \sum_{I,J} P_I^{ab\alpha\beta} P_J^{\beta\alpha cd} \int d^D q \delta((q+p_1)^2)\delta((q-p_2)^2)\cdot \\
&\qquad T^I(p_1,p_2,p_1+q,p_2-q) T^{*J}(p_2-q,p_1+q,p_3,p_4)\\
&= \frac{s^{\frac{D-4}{2}}}{2^{D+1}(2\pi)^{D-2}} \sum_I \int d\Omega P_I^{abcd} T^I(s,\eta_1) T^{*I}(s,\eta_2),
\end{aligned}
$$

where the orthogonality relation of projectors (III.2-20) was used. Here $\eta_{1,2}$ are cosines of scattering angles in c.m.s., $\eta_i = 1 + \frac{2t_i}{s_i}$. The integration over spherical coordinates goes over $(D-2)$ angles, i.e.

$$
d\Omega = \sin^{D-3}\theta \ \sin^{D-4}\varphi_1 \ ... \ \sin\varphi_{D-2} \ d\theta d\varphi_1..d\varphi_{D-3}.
$$

The cosines $\eta_{1,2}$ and η are related to each other by the simple trigonometrical expression, $\eta_2 = \eta\eta_1 + \sqrt{(1-\eta^2)(1-\eta_1^2)}\cos\varphi_1$.

The partial waves in the D dimensions are defined as

$$
\begin{aligned}
T^I(s,t) &= 64\pi \sum_{l=0}^{\infty} \frac{2l+D-3}{2} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} C_l^{\frac{D-3}{2}}(\eta) t_l^I(s) \qquad \text{(D.1-3)}\\
t_l^I(s) &= \frac{1}{64\pi} \int_{-1}^{1} (1-\eta^2)^{\frac{D-4}{2}} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} \frac{2^{D-4} l!}{\Gamma(l+D-3)} T^I(s,t) C_l^{\frac{D-3}{2}}(\eta) d\eta,
\end{aligned}
$$

where $C_l^{\nu}(z)$ are Gegenbauer polynomials. At $D=4$ this decomposition turns to the usual partial wave decomposition (III.1-7). Using the connection between η_1 and η_2 one can integrate out the φ_1-dependence in eqn. (D.1-2) with the help of the sum theorem for Gegenbauer polynomials

$$
\begin{aligned}
&\int_0^{\pi} C_l^{\frac{D-3}{2}}(\eta\eta_1 + \sqrt{(1-\eta^2)(1-\eta_1^2)}\cos\varphi) \sin^{D-4}\varphi d\varphi \\
&\qquad = 2^{D-4} \frac{l!\Gamma^2\left(\frac{D-3}{2}\right)}{\Gamma(l+D-3)} C_l^{\frac{D-3}{2}}(\eta) C_l^{\frac{D-3}{2}}(\eta_1).
\end{aligned}
$$

The integral over η_1 recalls the Gegenbauer orthogonality relation, and the integral over the residually angles gives the volume of S^{D-1}/S^{D-4}. Collecting all together one obtains the diagonal in l equation:

$$
\mathrm{Im} t_l^I(s) = \pi^{\frac{D-4}{2}} \left(\frac{s}{(4\pi)^2}\right)^{\frac{D-4}{2}} |t_l^I(s)|^2 + \mathcal{O}(\text{Inelastic part}) \qquad \text{(D.1-4)}
$$

The analytic continuation of the unitarity relation to the $s < 0$ area can be obtained using the dispersion relation. At the two-particles intermediate state approximation, the analytical properties of the amplitude in D dimensions are the same as in $D = 4$ case, namely the amplitude has the s-channel cut from $4m^2$ to $+\infty$, and the u-channel cut from $-\infty$ to 0. We have to switch on the masses in order to avoid the problems with the coalescing of branch points. The simple dispersion relation at fixed t with no subtractions can be presented in the form, e.g. [58],

$$T^I(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left(\frac{\delta^{II'}}{s'-s} + \frac{C_{su}^{II'}}{s'-4m^2+t+s}\right)\mathrm{Im}T^{I'}(s',t), \tag{D.1-5}$$

where the matrix C_{su} is the crossing matrix building up from projectors P_I, (III.2-25). To be sure that the dispersion relation converges one should make subtractions. But the subtractions do not influence the imaginary part of the amplitude, and we omit them. The discontinuity over the left-hand cut ($s < 0$) is given only by the second term in the brackets (D.1-5). In the partial waves basis (D.1-3) it reads

$$\mathrm{Im}\, t_l^I(s) = \sum_{l'=0}^{\infty} C_{su}^{II'} \frac{2^{D-3}(2l'+D-3)}{\Gamma(l+D-3)} \frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi} l! \int_{4m^2}^{4m^2-s} \frac{ds'}{s-4m^2} \cdot \tag{D.1-6}$$
$$\left[\frac{4s'(4m^2-s-s')}{(s-4m^2)^2}\right]^{\frac{D-4}{2}} C_l^{\frac{D-3}{2}}\left(\frac{s+2s'-4m^2}{4m^2-s}\right) C_{l'}^{\frac{D-3}{2}}\left(\frac{2s+s'-4m^2}{4m^2-s'}\right)\mathrm{Im}\, t_{l'}^{I'}(s').$$

Taking the limit $m^2 \to 0$ one obtains the expression

$$\begin{aligned}\mathrm{Im}\, t_l^I(s) &= \sum_{l'=0}^{\infty} C_{su}^{II'} \frac{2^{D-3}(2l'+D-3)}{\Gamma(l+D-3)} \frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi} l! \int_0^{-s} \frac{ds'}{s} \\ &\times \left[-4\frac{s'}{s}\left(1+\frac{s'}{s}\right)\right]^{\frac{D-4}{2}} C_l^{\frac{D-3}{2}}\left(\frac{s+2s'}{-s}\right) C_{l'}^{\frac{D-3}{2}}\left(\frac{2s+s'}{-s'}\right)\mathrm{Im}\, t_{l'}^{I'}(s').\end{aligned} \tag{D.1-7}$$

In $D = 4$ limit this relation was found for the case of $\pi\pi$=scattering amplitude in [22] as a consequence of the Roy equation. It has the form

$$\mathrm{Im}\, t_l^I(s) = \sum_{l'=0}^{\infty} C_{su}^{II'} \frac{2(2l'+1)}{s} \int_0^{-s} ds'\, P_l\left(\frac{s+2s'}{-s}\right) P_{l'}\left(\frac{2s+s'}{-s'}\right)\mathrm{Im}\, t_{l'}^{I'}(s'). \tag{D.1-8}$$

It is more convenient to rewrite the relation (D.1-7) through the integration over the scattering angle:

$$
\begin{aligned}
\operatorname{Im} t_l^I(s) &= -\sum_{l'=0}^{\infty} C_{su}^{II'} \frac{(2l'+D-3)}{\Gamma(l+D-3)} \frac{\Gamma^2\left(\frac{D-3}{2}\right)}{\pi} 2^{D-4} l! \qquad \text{(D.1-9)} \\
&\times \int_{-1}^{1} d\eta\, (1-\eta^2)^{\frac{D-4}{2}} (-1)^{l+l'} C_l^{\frac{D-3}{2}}(\eta) C_{l'}^{\frac{D-3}{2}}\left(\frac{\eta+3}{\eta-1}\right) \operatorname{Im} t_{l'}^{I'}\left(\frac{s}{2}(\eta-1)\right).
\end{aligned}
$$

E

Tables of LLog coefficients in various models

Here, we present the table of the LLog coefficients obtained from evaluation of eqn.(III.2-37) up to 4-loop order. The presentation of results is limited by the size of the page.

Table V-1: Table of isospin-0 LLog coefficients for the σ-model at $D = 4$, $\omega^0_{nl} \cdot (N-1)^{-1}$

$n \setminus l$	0	2	4
1	1		
2	$\frac{N}{2} - \frac{1}{9}$	$\frac{5}{18}$	
3	$\frac{N^2}{4} - \frac{61N}{144} + \frac{59}{144}$	$-\frac{13N}{144} + \frac{13}{48}$	
4	$\frac{N^3}{8} - \frac{631N^2}{2700} + \frac{46279N}{194400} - \frac{13309}{194400}$	$\frac{173N^2}{2160} - \frac{4313N}{38880} + \frac{5333}{38880}$	$\frac{N^2}{200} - \frac{49N}{5400} + \frac{8}{675}$
5	$\frac{N^4}{16} - \frac{136N^3}{675} + \frac{2498743N^2}{7776000} - \frac{3083771N}{11664000} + \frac{619889}{4665600}$	$-\frac{1417N^3}{40320} + \frac{481367N^2}{3628800} - \frac{727373N}{4082400} + \frac{1071107}{6531840}$	$-\frac{N^3}{280} + \frac{9787N^2}{756000} - \frac{449681N}{27216000} + \frac{81007}{5443200}$

Table V-2: Table of isospin-1 LLog coefficients for the σ-model at $D=4$, ω^1_{nl}

$n \setminus l$	1	3	5
1	1		
2	$-\frac{N}{2}+\frac{3}{2}$		
3	$\frac{9N^2}{40}-\frac{37N}{80}+\frac{49}{80}$	$\frac{N^2}{40}-\frac{37N}{720}+\frac{49}{720}$	
4	$-\frac{N^3}{10}+\frac{493N^2}{1200}-\frac{41791N}{64800}+\frac{8543}{12960}$	$-\frac{N^3}{40}+\frac{79N^2}{900}-\frac{7019N}{64800}+\frac{233}{2592}$	
5	$\frac{5N^4}{112}-\frac{52859N^3}{302400}+\frac{18963533N^2}{54432000}-\frac{9585587N}{27216000}+\frac{2685037}{10886400}$	$\frac{5N^4}{288}-\frac{98743N^3}{1555200}+\frac{4018577N^2}{34992000}-\frac{3612281N}{34992000}+\frac{292127}{4665600}$	$\frac{N^4}{2016}-\frac{20753N^3}{10886400}+\frac{363091N^2}{97977600}-\frac{17849N}{4898880}+\frac{101}{40824}$

Table V-3: Table of isospin-2 LLog coefficients for the σ-model at $D=4$, ω^2_{nl}

$n \setminus l$	0	2	4
1	-1		
2	$\frac{N}{3}+\frac{1}{9}$	$\frac{N}{6}-\frac{5}{18}$	
3	$-\frac{N^2}{8}+\frac{3N}{16}-\frac{59}{144}$	$-\frac{N^2}{8}+\frac{47N}{144}-\frac{13}{48}$	
4	$\frac{N^3}{20}-\frac{857N^2}{10800}+\frac{21131N}{194400}+\frac{13309}{194400}$	$\frac{N^3}{14}-\frac{449N^2}{2160}+\frac{68711N}{272160}-\frac{5333}{38880}$	$\frac{N^3}{280}-\frac{41N^2}{3600}+\frac{407N}{25200}-\frac{8}{675}$
5	$-\frac{N^4}{48}+\frac{1727N^3}{25920}-\frac{3323209N^2}{23328000}+\frac{1492651N}{11664000}-\frac{619889}{4665600}$	$-\frac{25N^4}{672}+\frac{112891N^3}{725760}-\frac{4774289N^2}{16329600}+\frac{612299N}{2041200}-\frac{1071107}{6531840}$	$-\frac{N^4}{224}+\frac{21797N^3}{1209600}-\frac{1747919N^2}{54432000}+\frac{282487N}{9072000}-\frac{81007}{5443200}$

Table V-4: Table of isospin-0 LLog coefficients for the ϕ^4-model at $D=6$, ω^0_{nl}

$n \setminus l$	0	2	4
1	$-(N+2)$		
2	$\frac{1}{6}(N+2)(N-1)$		
3	$-\frac{1}{36}(N+2)^2(N-1)$		
4	$(N+2)^2(N-1)(\frac{N}{216}+\frac{1}{540})$	$-\frac{7}{9720}(N+2)^2(N-1)$	
5	$-(N+2)^2(N-1)(\frac{N^2}{1296}+\frac{223N}{151200}+\frac{1}{226800})$	$-(N+2)^2(N-1)(\frac{23N}{388800}+\frac{101}{583200})$	$-(N+2)^2(N-1)(\frac{N}{680400}+\frac{23}{5103000})$

Table V-5: Table of isospin-1 LLog coefficients for the ϕ^4-model at $D=6$, ω^1_{nl}

$n \setminus l$	1	3	5
1	0		
2	$\frac{1}{18}(N+2)$		
3	$\frac{1}{108}(N+2)^2$		
4	$(N+2)^2(\frac{N}{756}-\frac{2}{2835})$	$(N+2)^2(\frac{N}{15120}-\frac{1}{28350})$	
5	$(N+2)^2(\frac{5N^2}{27216}+\frac{N}{4320}-\frac{53}{136080})$	$(N+2)^2(\frac{N^2}{45360}+\frac{N}{43200}-\frac{127}{2268000})$	

Table V-6: Table of isospin-2 LLog coefficients for the ϕ^4-model at $D=6$, ω^2_{nl}

$n \setminus l$	1	3	5
1	-2		
2	$-\frac{1}{6}(N+2)$		
3	$(N+2)(\frac{1}{18}-\frac{N}{60})$	$-\frac{N}{540}$	
4	$(N+2)(-\frac{N^2}{540}+\frac{N}{2700}-\frac{1}{270})$	$(N+2)(-\frac{N^2}{2160}-\frac{29N}{24300}+\frac{7}{4860})$	
5	$(N+2)(-\frac{N^3}{4536}+\frac{19N^2}{453600}-\frac{17N}{56700}+\frac{1}{113400})$	$(N+2)(-\frac{N^3}{11664}-\frac{77N^2}{388800}+\frac{N}{97200}+\frac{101}{291600})$	$(N+2)(-\frac{N^3}{408240}-\frac{N^2}{170100}$ $-\frac{N}{1701000}+\frac{23}{2551500})$

Bibliography

[1] S.Weinberg, Phisica A, **96**, 327-340, 1979.

[2] D. Diakonov, V. Y. Petrov and P. V. Pobylitsa, Nucl. Phys. B **306** (1988) 809.

[3] M. Praszalowicz and G. Valencia, Nucl. Phys. B **341** (1990) 27.

[4] S. Weinberg, Phys. Rev. **166** (1968) 1568.

[5] S. Scherer, Adv. Nucl. Phys. **27** (2003) 277 [arXiv:hep-ph/0210398].
S. Scherer, Prog. Part. Nucl. Phys. **64** (2010) 1 [arXiv:0908.3425 [hep-ph]].

[6] J. Bijnens, arXiv:0904.3713 [hep-ph].

[7] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[8] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[9] J. Gasser, M. A. Ivanov and M. E. Sainio, Nucl. Phys. B **745** (2006) 84 [arXiv:hep-ph/0602234].

[10] J. Bijnens, P. Dhonte and P. Talavera, JHEP **0405** (2004) 036 [arXiv:hep-ph/0404150].

[11] J. Bijnens, G. Colangelo and G. Ecker, JHEP **9902** (1999) 020 [arXiv:hep-ph/9902437].

[12] J. Bijnens, G. Colangelo and G. Ecker, Phys. Lett. B **441** (1998) 437 [arXiv:hep-ph/9808421].

[13] M. Buchler, Eur. Phys. J. C **44** (2005) 111 [arXiv:hep-ph/0504180].

[14] M. Buchler and G. Colangelo, Eur. Phys. J. C **32** (2003) 427 [arXiv:hep-ph/0309049].

[15] J. Bijnens and L. Carloni, Nucl. Phys. B **827** (2010) 237 [arXiv:0909.5086 [hep-ph]].

[16] J. Bijnens and L. Carloni, arXiv:1008.3499 [hep-ph].

[17] D. I. Kazakov and G. S. Vartanov, J. Phys. A **39** (2006) 8051 [arXiv:hep-th/0509208].

[18] D. Anselmi, JHEP **0507** (2005) 077 [arXiv:hep-th/0502237].

[19] G. Parisi, Nucl. Phys. B **100** (1975) 368.

[20] D. I. Kazakov and G. S. Vartanov, JHEP **0706** (2007) 081 [arXiv:0707.2564 [hep-th]].

[21] S. Antusch and M. Ratz, JHEP **0207** (2002) 059 [arXiv:hep-ph/0203027].

[22] M. Bissegger and A. Fuhrer, Phys. Lett. B **646**, 72 (2007) [arXiv:hep-ph/0612096].

[23] N. Kivel and M. V. Polyakov, Phys. Lett. B **664** (2008) 64 [arXiv:0707.2208 [hep-ph]].

[24] N. Kivel, M. V. Polyakov and A. Vladimirov, Phys. Rev. Lett. **101** (2008) 262001 [arXiv:0809.3236 [hep-ph]].

[25] N. A. Kivel, M. V. Polyakov and A. A. Vladimirov, JETP Lett. **89** (2009) 529 [arXiv:0904.3008 [hep-ph]].

[26] J. Koschinski, M. V. Polyakov and A. A. Vladimirov, Phys. Rev. D **82** (2010) 014014 [arXiv:1004.2197 [hep-ph]].

[27] N. Kivel, M. V. Polyakov and A. Vladimirov, Phys. Rev. D **79** (2009) 014028 [arXiv:0809.2064 [hep-ph]].

[28] M. V. Polyakov and A. Vladimirov, in preparation.

[29] J.C.Collins, *Renormalization* (Cambridge University Press, 1984)

[30] N.N. Bogoliubov, D.V. Shirkov *Introduction to Theory of Quantized Fields*, (John Wiley & Sons Inc, 1959)

[31] S. Weinberg, arXiv:0908.1964 [hep-th].

[32] A.N.Vasil'ev *The field theoretic renormalization group in critical behavior theory and stochastic dynamics* (Boca Raton: Chapman & Hall/CRC, 2004)

[33] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.
Y. Nambu and G. Jona-Lasinio, Phys. Rev. **124** (1961) 246.

[34] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[35] T. Becher and H. Leutwyler, Eur. Phys. J. C **9** (1999) 643 [arXiv:hep-ph/9901384].

[36] E. E. Jenkins and A. V. Manohar, Phys. Lett. B **255** (1991) 558.

[37] E. E. Jenkins, A. V. Manohar and M. B. Wise, Phys. Rev. Lett. **75** (1995) 2272 [arXiv:hep-ph/9506356].

[38] A. Dobado, M. J. Herrero and T. N. Truong, Phys. Lett. B **235** (1990) 134.

[39] H. Leutwyler, arXiv:hep-ph/9406283.

[40] V. Bernard, N. Kaiser and U. G. Meissner, Int. J. Mod. Phys. E **4** (1995) 193 [arXiv:hep-ph/9501384].

[41] K. Symanzik, Commun. Math. Phys. **18** (1970) 227.

[42] G. Colangelo, Phys. Lett. B **350** (1995) 85 [Erratum-ibid. B **361** (1995) 234] [arXiv:hep-ph/9502285].

[43] S. R. Coleman, R. Jackiw and H. D. Politzer, Phys. Rev. D **10** (1974) 2491.

[44] A. Dobado and J. R. Pelaez, Phys. Lett. B **286** (1992) 136.

[45] A. Dobado, A. L. Maroto and J. Morales, Nuovo Cim. A **108** (1995) 335 [arXiv:hep-ph/9401230].

[46] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Phys. Lett. B **374** (1996) 210 [arXiv:hep-ph/9511397].

[47] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Nucl. Phys. B **508** (1997) 263 [Erratum-ibid. B **517** (1998) 639] [arXiv:hep-ph/9707291].

[48] J. Bijnens, G. Colangelo and P. Talavera, JHEP **9805** (1998) 014 [arXiv:hep-ph/9805389].

[49] J. F. Donoghue and H. Leutwyler, Z. Phys. C **52** (1991) 343.

[50] D. I. Kazakov and G. S. Vartanov, JHEP **0706** (2007) 081 [arXiv:0707.2564 [hep-th]].
D. I. Kazakov and G. S. Vartanov, arXiv:hep-th/0607177.

[51] E. Witten, Nucl. Phys. B **160** (1979) 57.

[52] S.S.Cheng, W.Li *Analytic solutions of functional equations* (World Scientific Publishing Co., 2008)

[53] A.D.Polyanin, A.V.Manzhirov *Handbook of integral equations: Second edition* (Chapman and Hall/CRC; 2 edition, 2008)

[54] P.D.B.Collins, E.J.Squires *Regge poles in particle physics* (Springer, 1968) P.D.B.Collins,*An Introduction to Regge Theory and High Energy Physics* (Cambridge University Press; 1 edition, 2009)

[55] R.J.Eden, P.V.Landshof, D.I.Olive, J.C.Polkinghorne *The analytic S-matrix* (Cambridge University Press, 1966)

[56] R. Omnes, Nuovo Cim. **8** (1958) 316.

[57] P.Cvitanovic, *Group Theory: Birdtracks, Lie's, and Exceptional Groups* (Princeton University Press, 2008)

[58] S. M. Roy, Phys. Lett. B **36**, 353 (1971).

[59] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).

[60] O. N. Soldatenko, A. N. Vall and A. A. Vladimirov, Eur. Phys. J. A **38** (2008) 71 [arXiv:0805.2296 [hep-ph]].

[61] J. C. Collins, D. E. Soper and G. Sterman, Adv. Ser. Direct. High Energy Phys. **5** (1988) 1 [arXiv:hep-ph/0409313].

[62] N. Kivel and M. V. Polyakov, arXiv:hep-ph/0203264.
N. Kivel and M. V. Polyakov, Nucl. Phys. A **711** (2002) 211.

[63] A. V. Belitsky and A. V. Radyushkin, Phys. Rept. **418** (2005) 1 [arXiv:hep-ph/0504030].

[64] M. Diehl, Phys. Rept. **388** (2003) 41 [arXiv:hep-ph/0307382].

[65] K. Goeke, M. V. Polyakov and M. Vanderhaeghen, Prog. Part. Nucl. Phys. **47** (2001) 401 [arXiv:hep-ph/0106012].

[66] F.J.Yndurain *Quantum Chromodynamics: An Introduction to the Theory of Quarks and Gluons* (Springer-Verlag, 1983)

[67] D. Müller, D. Robaschik, B. Geyer B., F.-M. Dittes, and J. Horejsi, Fortschr. Phys. **42**, 101 (1994).

[68] A.V. Radyushkin, Phys. Rev. D **58**, 114008 (1998).

[69] M.V. Polyakov , and C. Weiss, Phys. Rev. D **60**, 114017 (1999).

[70] M.V. Polyakov, Nucl. Phys. B **555** (1999) 231 [arXiv:hep-ph/9809483].

[71] M. Diehl, A. Manashov and A. Schafer, Phys. Lett. B **622** (2005) 69 [arXiv:hep-ph/0505269].

[72] A.V. Radyushkin, Phys. Rev. D **56**, 5524 (1997).

[73] M. Burkardt, Phys. Rev. D **62** (2000) 071503 [Erratum-ibid. D **66** (2002) 119903] [arXiv:hep-ph/0005108].

[74] M. Strikman and C. Weiss, Phys. Rev. D **69** (2004) 054012 [arXiv:hep-ph/0308191].

[75] K. Wijesooriya, P. E. Reimer and R. J. Holt, Phys. Rev. C **72** (2005) 065203 [arXiv:nucl-ex/0509012].

[76] V.A.Smirnov *Feynman Integral Calculus* (Springer Berlin Heidelberg, 2006)

[77] H.Bateman, A.Erdelyi *Higher transcendental functions* (McGraw Hill Book Company, 1953)

[78] M.I.Nechepurenko *Iterations of real functions and functional equations* (Novosibirsk, 1993) [in Russian]

[79] F.W.J.Olver *Asymptotics and Special functions* (Academic Press Inc., 1974)

[80] N.J.Vilenkin *Special functions and the Theory of group representations* (American Mathematical Society; Revised edition, 1968)

[81] D. I. Kazakov, O. V. Tarasov and A. A. Vladimirov, Sov. Phys. JETP **50**, 521 (1979) [Zh. Eksp. Teor. Fiz. **77**, 1035 (1979)].

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