Evolution of transverse momentum dependent distributions

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Introduction

The talk is a mini-review about

- transverse momentum dependent (TMD) factorization theorems,
- soft factors,
- rapidity divergences,
- rapidity anomalous dimension and TMD evolution,
- and its interpretation.

Plan of the talk

- introduction to TMD factorization,
- TMD soft factor and rapidity divergences,
- renormalization theorem for rapidity divergences,
- evolution equation and ζ-prescription
- comparison with the data
- interpretation of rapidity anomalous dimension

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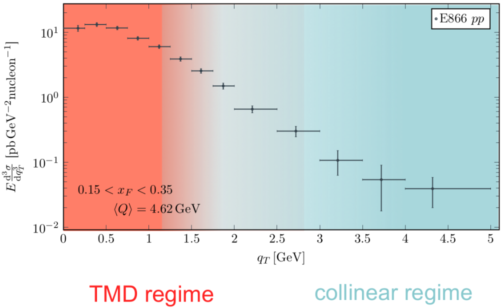
Transverse momentum dependent (TMD) factorization describes double-inclusive processes in the regime of small transverse momentum ($q_T^2 \ll Q^2$)

processes: $h_1 + h_2 \to \gamma^*/Z/W + X$ "Drell-Yan"

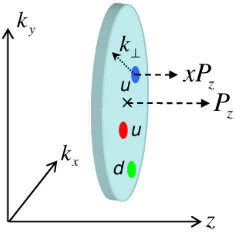
$h_1 + \gamma^* \to h_2 + X$ semi-inclusive DIS (SIDIS)

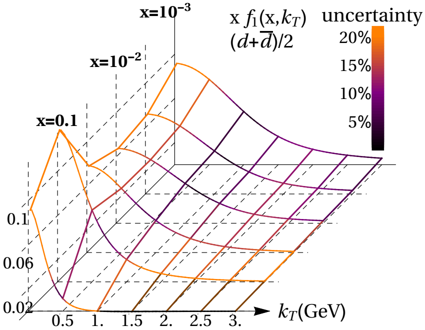
$e^+e^- \to h_1 + h_2 + X$



The transverse momentum of photon q_T is determined with respect to "hadron plane"

In TMD regime the produced transverse momentum is mostly of "non-perturbative" origin: $\Rightarrow$ TMD distributions (PDFs and FFs)





[Bertone,Scimemi,AV,1902.08474]

TMD distributions should not be mistaken with collinear distributions (although they have some common points).

- Structurally different: different divergences and different evolution.

Part I:
TMD factorization

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim L_{\mu\nu}\int d^4x\, e^{iqx}\sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

Field modes factorization

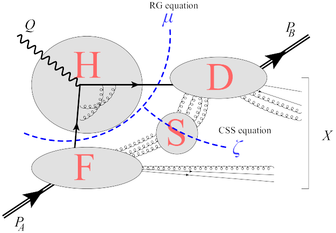
TMD soft factor

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\ \ \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T) + Y$$

power suppressed terms

TMD FF

TMD PDF



Structure of TMD factorization

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim L_{\mu\nu}\int d^4x\,e^{iqx}\sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

Field modes factorization

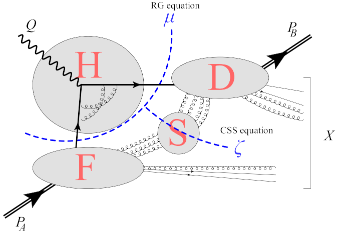
$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

Summation of soft gluon exchanges $\Rightarrow$ Wilson lines

$$[x,y] = P\exp\left(ig\int_x^y dz^\mu A_\mu(z)\right)$$

Parallel transporter of gluon field.

- Sums soft-exchanges between hadron and parton
- Present in all elements of factorization theorems

Example: parton distribution function

$$f(x) = \int \frac{d\lambda}{2\pi} e^{ixp\lambda} \langle \text{hadron}|\bar{q}(\lambda n)[\lambda n, 0]q(0)|\text{hadron}\rangle$$

$n^2 = 0$
on light-cone

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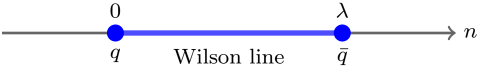
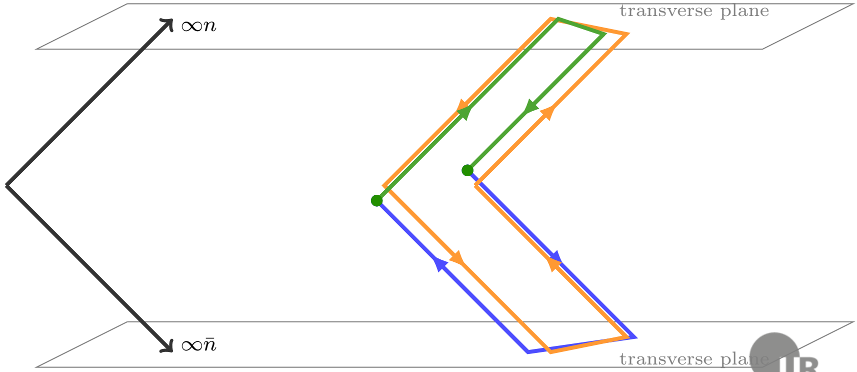


$n^2 = 0$
on light-cone

TMD factorization is full of (light-like) Wilson lines

$$\frac{d\sigma}{dX} \simeq H(Q) \int \frac{d^2 b}{(2\pi)^2} e^{i(bk)_T} \; f(x_A, b) \; S(b) \; f(x_B, b)$$



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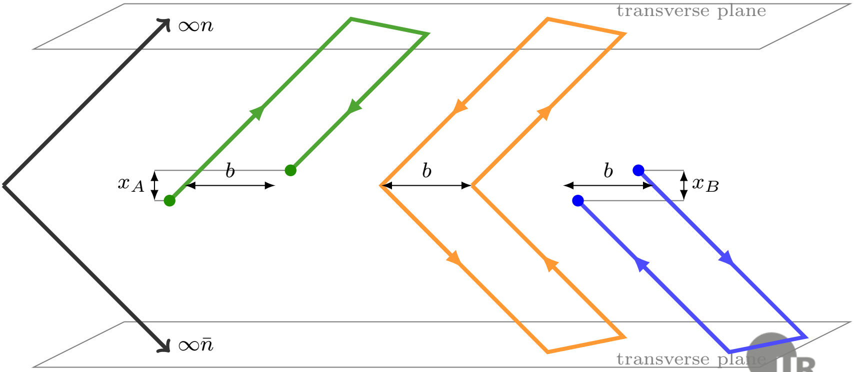
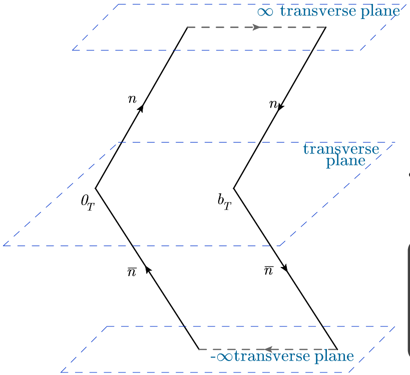


TMD soft factor

$$S(\mathbf{b}_T) = \langle 0 | \mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n \cdot \bar{n}) = 1$$

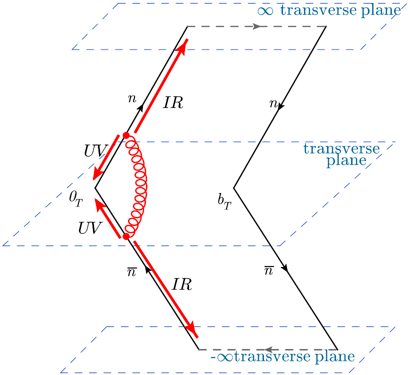
Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Looks simple, but SF is a theoretician's nightmare.
Multiple divergences!

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-}$$

$$= \int_0^\infty \frac{dx^+}{x^+} \int_0^\infty \frac{dy^-}{y^-}$$

$$= (\mathrm{UV}+\mathrm{IR})(\mathrm{UV}+\mathrm{IR})$$

Some people set it to zero.

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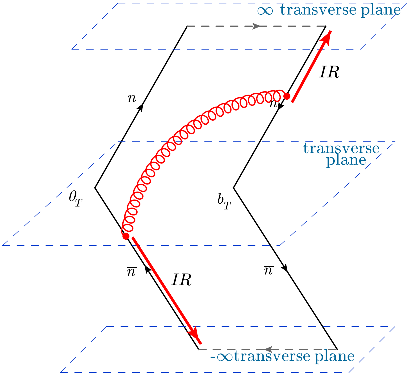
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned}\int \quad dxdy \quad & D(x-y)\\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}\\ = \quad & \text{IR at } x, y \to \infty\end{aligned}$$

However, it exactly cancels IR from the previous diagram
Proved at all orders,
e.g.[Echevarria,Scimemi,AV,1511.05590]

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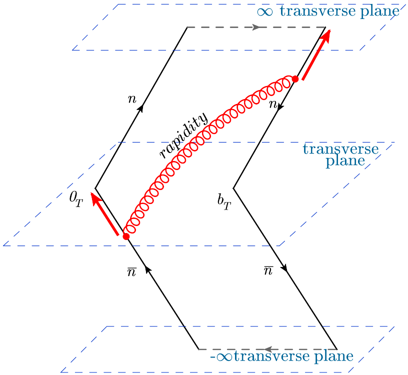
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\boldsymbol{\Phi}_n(\mathbf{0}_T)\boldsymbol{\Phi}_n^\dagger(\mathbf{b}_T)\boldsymbol{\Phi}_{-n}(\mathbf{b}_T)\boldsymbol{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\int dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \underbrace{\int_0^\infty \frac{d\sigma}{\sigma}}_{\text{rap.div}} \underbrace{\int_0^\infty \frac{dLL}{(2L^2+\mathbf{b}^2)}}_{\text{IR}}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

Regularizations for rapidity divergences

- Rapidity divergences are not regularized by dim.reg.
- There are many regularizations:
  - δ-regularization [Echevarria,Scimemi,AV,1511.05590],
  - exponential-regularization [Li,Neill,Zhu,1604.00392],
  - off-light-cone Wilson lines [Collins' textbook],
  - analytical regularization [Chiu, et al,1104.0881],
  - ...

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$
(2-loop check [1511.05590])

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

It allows to split rapidity divergences and define individual TMDs.

- Important note 1: the structure holds for arbitrary ϵ
- Important note 2: the structure holds at all orders of PT [AV,1707.07606]

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Factorization of rapidity-divergences

$$S(b_T) = \exp\big(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\big)$$

$$= \exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\zeta^+(\delta^+)^2) + \frac{B(b_T,\epsilon)}{2}\right)\exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\zeta^-(\delta^-)^2) + \frac{B(b_T,\epsilon)}{2}\right)$$

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rap.div. in n-direction

rap.div. in $\bar{n}$-direction

- The rapidity divergences related to different sectors factorize
- Factorization introduces an additional scales ζ: (here $\zeta^+\zeta^- = 1$)
- The factorization of rapidity divergences in TMD soft-factor is a consequence of **renormalization theorem for rapidity divergences.**

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Renormalization theorem for rapidity divergences.[AV,1707.07606]

At *any finite order* of perturbation theory there exists the "rapidity divergence renormalization factor" $\mathbf{R}_n$, which contains only rapidity divergences associated with the direction n, such that the combination

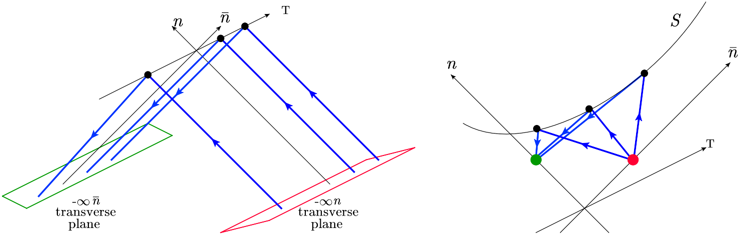
$$S^R(\{b\}, \zeta^+, \zeta^-) = \mathbf{R}_n(\{b\}, \zeta^+) S(\{b\}) \mathbf{R}_n^\dagger(\{b\}, \zeta^-)$$

is free of rapidity divergences.

The essence of proof is the equivalence of rapidity divergences and ultra-violet divergences.

In conformal field theory

$\mathbf{S}(b) \xleftrightarrow[\text{tranformation}]{\text{conformal}} \mathbf{\Omega}(v)$



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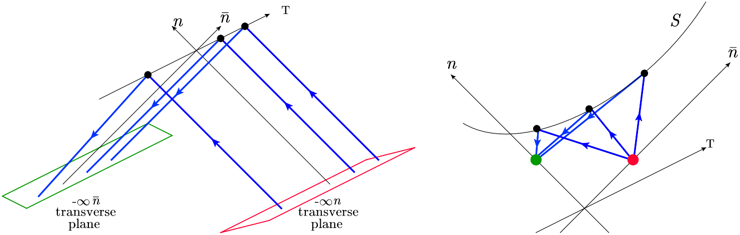
In conformal field theory

$$\mathbf{R}(b)\mathbf{S}(b)\mathbf{R}^\dagger(b) \xleftrightarrow[\text{tranformation}]{\text{conformal}} \mathbf{Z}(v)\mathbf{\Omega}(v)\mathbf{Z}^\dagger(v)$$



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In conformal field theory

finite $\qquad\qquad$ finite

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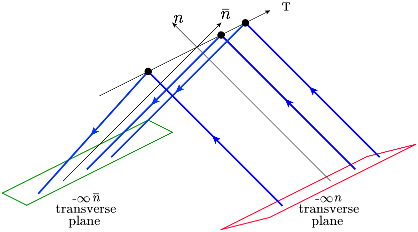

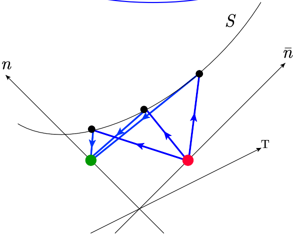

In CFT rapidity renormalization factor equals to UV renormalization

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In QCD

The existence of renormalization can be proved order-by-order with iterations using

- Renormalization statement in CFT
- Conformal-invariance of QCD at tree-order.

In QCD

The existence of renormalization can be proved order-by-order with iterations using

- Renormalization statement in CFT
- Conformal-invariance of QCD at tree-order.

Similarly to UV renormalization the rapidity-divergence renormalization satisfies **renormalization group equation** with respect to rapidity divergence renormalization scale ζ. The scaling with ζ has anomalous dimension

$$\mathcal{D}(b) = \frac{1}{2}\mathbf{R}^{-1}(b,\zeta)\frac{d}{d\ln\zeta}\mathbf{R}(b,\zeta)$$

In literature, this object is known

- under different names: "non-perturbative Sudakov kernel", "CSS kernel", "rapidity anomalous dimension".
- under different letters $-K(b)/2$ [Collins,et al], $F_{q\bar{q}}(b)$ [Becher,Neubert], γ_ν, ...

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Important consequence:
correspondence between soft- and rapidity-anomalous dimensions.

In conformal field theory

$$\mathcal{D}(\mu, \mathbf{b}) = \gamma_s(\mu, (v_1 \cdot v_2)), \qquad (v_1 \cdot v_2) = \mathbf{b}^2 e^{2\gamma_E}/4$$

Checked by explicit calculation in $\mathcal{N} = 4$SYM [Li,Zhu,1604.01404]

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In QCD

In QCD the same relation holds at the critical point (e.g.$\epsilon^* = -\beta(a_s)$)

$$\mathcal{D}(\mu, \mathbf{b}; \epsilon^*) = \gamma_s(\mu, (v_1 \cdot v_2)), \qquad (v_1 \cdot v_2) = \mathbf{b}^2 e^{2\gamma_E}/4$$

This relation allows one to gain β-function terms of higher-order from lower order [AV,1610.05791].

Using this relation, one can derive 3-loop $\mathcal{D}$ from 2-loop $\mathcal{D}$ and 3-loop γ_s.

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

How to use it?

- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$
- $\epsilon^* = 0 - a_s\beta_0 - a_s^2\beta_1 - a_s^3\beta_2 - ....$
- We can compare order by order in PT

$$\begin{aligned}
\mathbf{D}_1(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_1(\{v\}),\\
\mathbf{D}_2(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_2(\{v\}) + \beta_0\mathbf{D}_1'(\{b\}),\\
\mathbf{D}_3(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_3(\{v\}) + \beta_0\mathbf{D}_2'(\{b\}) + \beta_1\mathbf{D}_1'(\{b\}) - \frac{\beta_0^2}{2}\mathbf{D}_1''(\{b\}),
\end{aligned}$$

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TMD rapidity anomalous dimension

3-loop expression for RAD

$$\mathcal{D}_1(\mathbf{b}^2,\epsilon) = -2a_s C_F\Big[\Big(\frac{\mathbf{b}^2}{4}\Big)^{\epsilon}\Gamma(-\epsilon)+\frac{1}{\epsilon}\Big] = a_s C_F\Big\{2\mathbf{L}_\mu + \epsilon\,\underbrace{(\mathbf{L}_\mu^2+\zeta_2)}_{D_1'} + ...\Big\}$$

$$\begin{aligned}\mathcal{D}_2(\mathbf{b}^2,\epsilon) &= a_s^2 C_F\Big\{\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)\\ &\quad+\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\Big(\frac{3(4-3\epsilon)}{2\epsilon}C_A - N_f\Big)\Big) + \boldsymbol{B}^{\epsilon}\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0 + \frac{\beta_0}{2\epsilon^2} - \frac{\Gamma_1}{2\epsilon}\Big\}\end{aligned}$$

Taking

$$\gamma_s = C_F a_s(\Gamma_0\mathcal{L}_\mu - \tilde{\gamma}_0) + C_F a_s^2(\Gamma_1\mathcal{L}_\mu - \tilde{\gamma}_1) + C_F a_s^3(\Gamma_2\mathcal{L}_\mu - \tilde{\gamma}_2) + ...$$

We find

$$\mathcal{D}_3(\mathbf{b}^2,0) = \text{logs} - \frac{\tilde{\gamma}_2}{2} + (\beta_1+\beta_0\Gamma_1)\zeta_2 - \frac{2}{3}\beta_0^2\zeta_3 + \beta_0\Big\{C_A\Big(\frac{2428}{81}-26\zeta_4\Big) - N_f\frac{328}{81}\Big\}$$

It coincides with the direct calculation [Li,Zhu,1604.01404].

$$\mathcal{D}^{(3)}_{L=0} = -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3-\frac{88}{3}\zeta_2\zeta_3-192\zeta_5-\frac{297029}{729}+\frac{6392}{81}\zeta_2+\frac{154}{3}\zeta_4\right)$$
$$-\frac{C_AN_f}{2}\left(-\frac{904}{27}\zeta_3+\frac{62626}{729}-\frac{824}{81}\zeta_2+\frac{20}{3}\zeta_4\right)-$$
$$\frac{C_FN_f}{2}\left(-\frac{304}{9}\zeta_3+\frac{1711}{27}-16\zeta_4\right)-\frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3-\frac{1856}{729}\right)$$

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Important

in QCD

rapidity anomalous dimension is

generically non-perturbative

- Non-perturbative terms important at $b \gtrsim \Lambda_{\text{QCD}}^{-1}$
- At $b \to 0$ is entirely perturbative
- There is no "non-perturbative" proof of factorization, but it is expected (e.g. b^2−correction factorizes at LO [Scimemi,AV,1609.06047], at all orders [AV,in prep.])
- All non-perturbative correction must turn to zero at ϵ^*, "renomalon nature".

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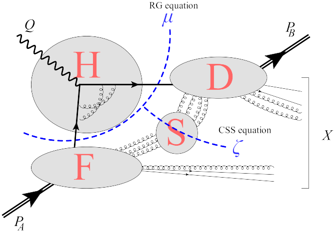
Final form of TMD factorization

$$d\sigma \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2) \quad \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

spliting rapidity singularities
$S(b_T) \to R(b_T, \zeta^+) \cdot S_0 \cdot R^\dagger(b_T, \zeta^-)$

$$d\sigma \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\; F(z_1, b_T; \zeta^+)\; D(z_2, b_T; \zeta^-) + Y$$



TMD PDF	TMD FF
$R\Phi_{h_1}$	$R\Delta_{h_2}$
(regular)	(regular)

S_0 is the finite parts
of rapidity renormalization
Commonly used
renormalization scheme: $S_0 = 1$

Part II:
TMD evolution

$$\frac{d\sigma}{dy dQ^2 d^2\mathbf{q}_T} = \sigma_0 \int d^2 b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1, b; \mu, \zeta_1) D_{f'\leftarrow h}(x_2, b; \mu, \zeta_2) + \ldots$$

Evolution

TMD evolution is given by 2 equations

$$\mu^2 \frac{dF(x,b;\mu,\zeta)}{d\mu^2} = \gamma_F(\mu,\zeta) F(x,b;\mu,\zeta), \qquad \zeta \frac{dF(x,b;\mu,\zeta)}{d\zeta} = -\mathcal{D}(\mu,b) F(x,b;\mu,\zeta)$$

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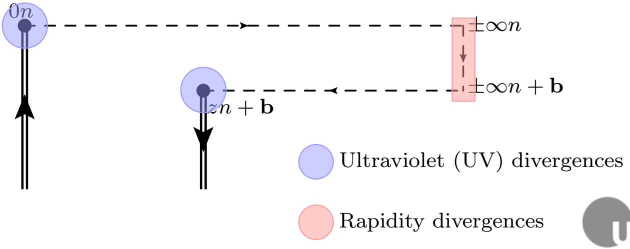
TMD operator



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TMD evolution is a double-scale evolution

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

Both anomalous dimensions related to each other (CS equation [Collins,Sopper,1981])

$$-\zeta \frac{d\gamma_F(\mu,\zeta)}{d\zeta} = 2\mu^2 \frac{d\mathcal{D}(\mu,b)}{d\mu^2} = \Gamma_{\text{cusp}}(\mu) \quad (3)$$

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$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

Solution: $F(x,\mathbf{b};\mu_f,\zeta_f) = R[\mathbf{b};(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,\mathbf{b};\mu_i,\zeta_i)$

Expression for R is known as "Sudakov exponent"

$$\times \exp\left\{ \ln \frac{\sqrt{\zeta_A}}{\mu_b} \tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_D(g(\mu');1) - \ln \frac{\sqrt{\zeta_A}}{\mu'} \gamma_K(g(\mu')) \right] \right\}. \quad (13.70)$$

This is probably the best formula for calculating and fitting TMD fragmentation functions;

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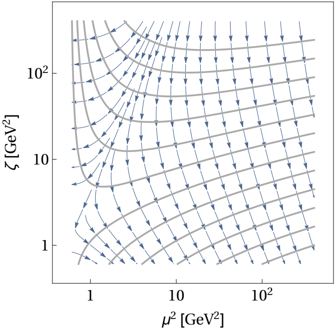
Two-dimensional picture



TMD evolution is 2D evolution

$$\mu^2 \frac{dF(x,b;\mu,\zeta)}{d\mu^2} = \gamma_F(\mu,\zeta)F(x,b;\mu,\zeta)$$

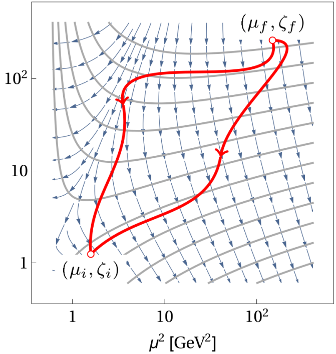
$$\zeta \frac{dF(x,b;\mu,\zeta)}{d\zeta} = -\mathcal{D}(\mu,b)F(x,b;\mu,\zeta)$$

or

$$\vec{\boldsymbol{\nabla}} F = \vec{\mathbf{E}} F$$

Evolution field	$\mathbf{E} = \left(\frac{\gamma_F}{2}, -\mathcal{D}\right)$
is conservative	$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$
Evol.potential:	$\mathbf{E} = \boldsymbol{\nabla} U$

Two-dimensional picture



TMD evolution is 2D evolution

$$R[\mathbf{b}; i \to f] =$$

$$\exp \int_P d\boldsymbol{\nu} \cdot \mathbf{E} = \exp(U_f - U_i) =$$

$$\exp \left[\int_P \left(\gamma_F(\mu, \zeta) \frac{d\mu}{\mu} - \mathcal{D}(\mu, \mathbf{b}) \frac{d\zeta}{\zeta} \right) \right]$$

- Path independence
- Unified picture of various evolution scenarios

Evolution field	$\mathbf{E} = \left(\frac{\gamma_F}{2}, -\mathcal{D} \right)$
is conservative	$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$
Evol.potential:	$\mathbf{E} = \boldsymbol{\nabla} U$

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Solution

$$F(x,\mathbf{b};\mu_1,\zeta_1)=R[\mathbf{b};(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)]F(x,\mathbf{b};\mu_2,\zeta_2)$$

Initial scales:

$\mu_1 \simeq Q$

$\zeta_1 = Q^2$

$\longrightarrow$

Final scales:

$\mu_2 \sim\ ??$

$\zeta_2 \sim\ ??$

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Solution

$$F(x, \mathbf{b}; \mu_1, \zeta_1) = R[\mathbf{b}; (\mu_1, \zeta_1) \to (\mu_2, \zeta_2)] F(x, \mathbf{b}; \mu_2, \zeta_2)$$

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$\longrightarrow$

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$\mu_2 \sim$??
$\zeta_2 \sim$??

Main complication:
$\mathbf{b}^2 \in (0, \infty)$
perturbative logarithms $\ln(\mathbf{b}^2 \mu^2)$, $\ln(\mathbf{b}^2 \zeta)$, $\ln(\mu^2/\zeta)$

Solution

$$F(x,\mathbf{b};\mu_1,\zeta_1) = R[\mathbf{b};(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)]F(x,\mathbf{b};\mu_2,\zeta_2)$$

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perturbative logarithms $\ln(\mathbf{b}^2\mu^2)$, $\ln(\mathbf{b}^2\zeta)$, $\ln(\mu^2/\zeta)$

Approaches

- CSS [Collins,Soper,Sterman,1984]

$$\mu = \mu^*(\mathbf{b}) \sim \begin{cases} 1/\mathbf{b} & \mathbf{b}\to 0, \\ \text{const} & \mathbf{b} \gg 0, \end{cases} \qquad \zeta = \mu^2$$

  - Formally, no large-logarithms in perturbative regeme
  - Extremely unstable perturbative expression
  - Non-perturbative model is dependent on the order of evolution
  - Commonly used LO approach, but useless at NNLO (unless very-high energies)

Solution

$$F(x,\mathbf{b};\mu_1,\zeta_1) = R[\mathbf{b};(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)]F(x,\mathbf{b};\mu_2,\zeta_2)$$

Initial scales:
$\mu_1 \simeq Q$
$\zeta_1 = Q^2$

$\longrightarrow$

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$\mu_2 \sim$??
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Main complication:
$\mathbf{b}^2 \in (0,\infty)$
perturbative logarithms $\ln(\mathbf{b}^2\mu^2)$, $\ln(\mathbf{b}^2\zeta)$, $\ln(\mu^2/\zeta)$

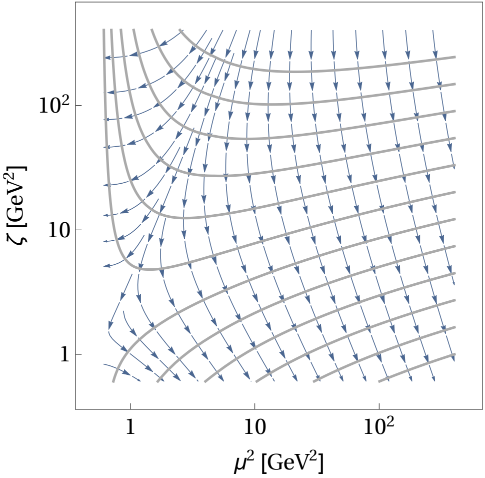
Approaches

- ζ-prescription [I.Scimemi,AV,1803.11089]

$$\mu = \textbf{doesn't matter}, \qquad \zeta = \zeta_\mu$$

  - Formally, no large-logarithms in perturbative regeme
  - Stable perturbative expression
  - Universal definition
  - Let $\mu \simeq Q$

TMD distribution is not defined by a scale (μ, ζ)
It is defined by an equipotential line.



The scaling is defined by
~~a difference between scales~~
a difference between potentials

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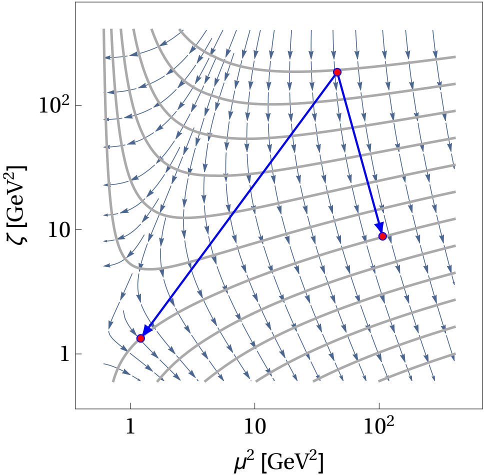
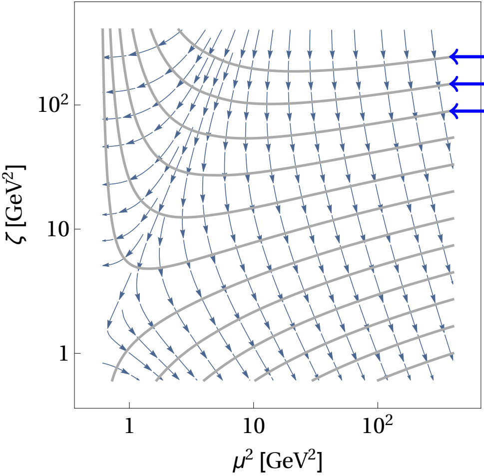


The scaling is defined by
~~a difference between scales~~
a difference between potentials

Evolution factor to both points
is the same
although the scales are
different by $10^2\mathrm{GeV}^2$

TMD distributions on the same equipotential line are equivalent.



$\mathrm{TMD}(x, b, 1)$

$\mathrm{TMD}(x, b, 2)$

$\mathrm{TMD}(x, b, 3)$

We can enumerate them by a lines not by (μ, ζ)

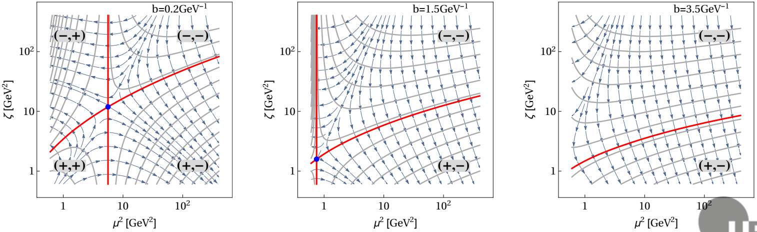
$$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$$

There is a unique line which passes though all μ's

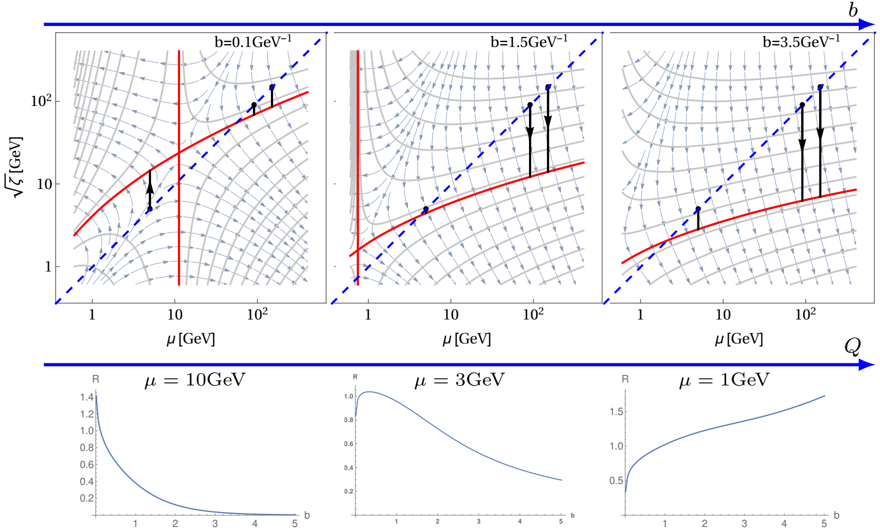
The optimal TMD distribution

$$F(x,b) = F(x,b;\mu,\zeta_\mu)$$

where ζ_μ is the special line.



The evolution potential depends on b.
Relative position of its elements (saddle-point, special lines) dictates the shape of evolution factor.



$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T} = \sigma_0 \int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1,b;\mu,Q^2) F_{f'\leftarrow h}(x_2,b;\mu,Q^2) + \dots$$

Evolution

$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T} = \sigma_0 \int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) R[\mathbf{b};(\mu,Q^2)\to \text{s.l.}]^2 F_{f\leftarrow h}(x_1,b) F_{f'\leftarrow h}(x_2,b) + \dots$$

Evolution factor has simple expression

$$R[\mathbf{b};(\mu,\zeta)\to \text{s.l.}] = \left(\frac{\zeta}{\zeta_\mu}\right)^{-\mathcal{D}(\mathbf{b},\mu)}$$

Good PT convergence
$\mu = Q$

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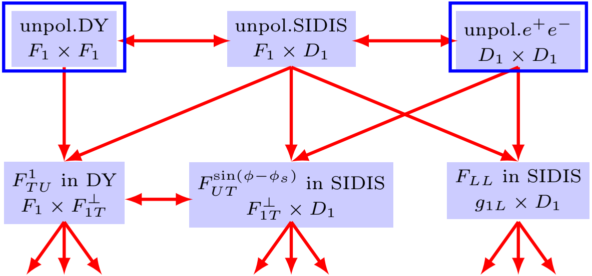
Part III:
Practice

Why unpolarized TMDPDF especially important?

Each TMD factorized cross-section has three NP functions

$$\frac{d\sigma}{dp_T^2 dQ} \simeq \sigma_0(Q) \int d^2\mathbf{b} e^{i\mathbf{b}\mathbf{p}_T} R[Q \to (\mu,\zeta)]^2 F_1(x_1,\mathbf{b};\mu,\zeta) F_2(x_2,\mathbf{b};\mu,\zeta)$$

- Two TMD distributions F_1 & F_2
- non-perturbative evolution $R \sim \exp(-\mathcal{D})$



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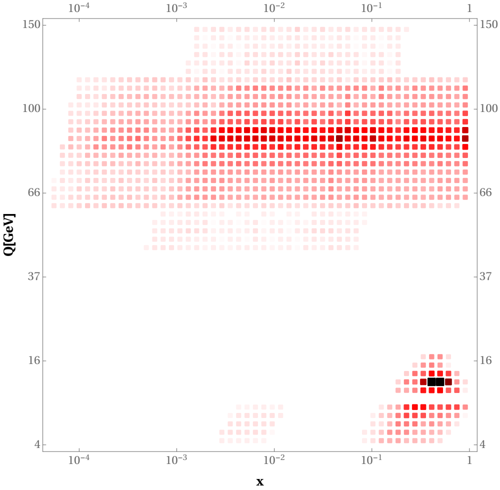
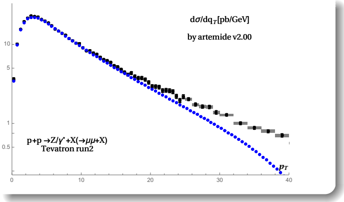
TMD factorization $\Rightarrow$ small-q_T/Q

$$\frac{q_T}{Q} < 0.25 \qquad \& \qquad \left(\frac{q_T}{Q}\right)^2 < \frac{\delta\sigma_{\text{uc}}}{\sigma}$$

dedicated study in [I.Scimemi,AV,1706.01473]



$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y} \sqrt{1 + \frac{q_T^2}{Q^2}}$$



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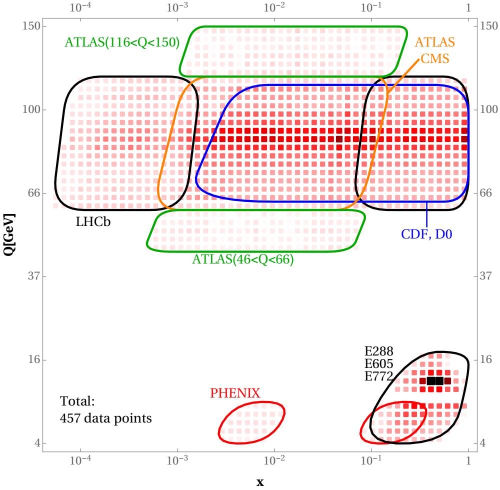


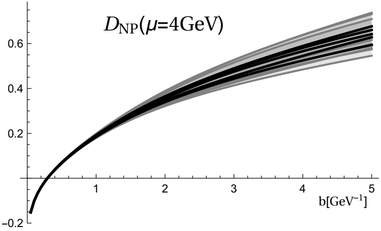
$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y} \sqrt{1 + \frac{q_T^2}{Q^2}}$$

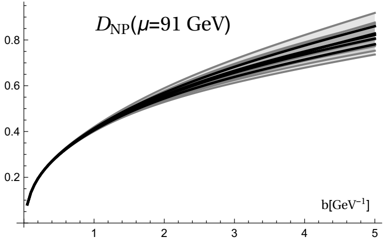
High-energy: CDF, D0, ATLAS, CMS, LHCb
194 points

Low-energy: E288, E605, E772, PHENIX
263 points

Total: 457 points
$4 < Q < 150\text{GeV}$
$x > 10^{-4}$

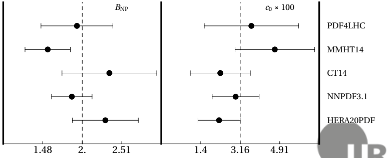
Non-perturbative evolution kernel





$$\mathcal{D}(\mathbf{b}) = \mathcal{D}_{\text{pert}}(b^*(\mathbf{b})) + c_0 \mathbf{b} b^*(\mathbf{b}), \qquad b^*(\mathbf{b}) = \mathbf{b}/\sqrt{1+\mathbf{b}^2/B_{\text{NP}}^2}$$

$$\boxed{\begin{array}{c} B_{\text{NP}} \simeq 2\text{GeV} \\ c_0 \simeq 0.03\text{GeV}^2 \end{array}}$$



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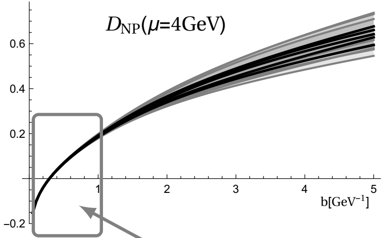
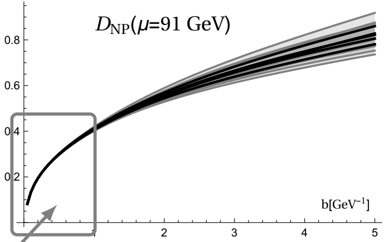
Non-perturbative evolution kernel

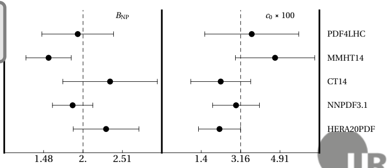




$$\mathcal{D}(\mathbf{b}) = \mathcal{D}_{\text{pert}}(b^*(\mathbf{b})) + c_0 \mathbf{b} b^*(\mathbf{b}), \qquad b^*(\mathbf{b}) = \mathbf{b}/\sqrt{1+\mathbf{b}^2/B_{\text{NP}}^2}$$

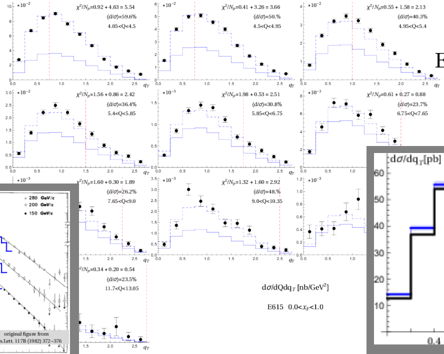
$B_{\text{NP}} \simeq 2$
$c_0 \simeq 0.03\text{GeV}^2$

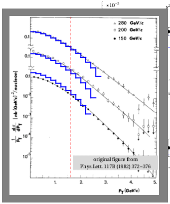
LHC
sensitive region

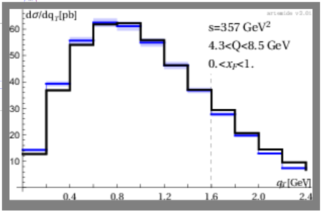


Extracted non-perturbative rapidity anomalous dimension
is universal and can be used to describe different data

Pion-induced Drell-Yan [AV,1907.10356]







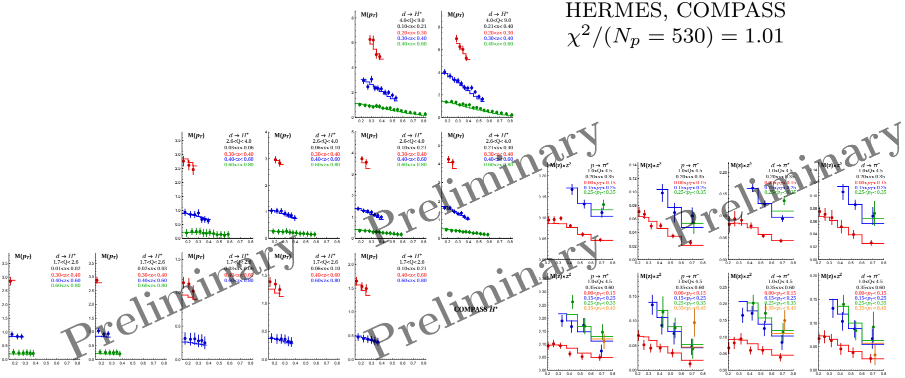
E537, E615, NA3
$\chi^2/(N_p = 80) = 1.44$
E615 normalization issue?

Extracted non-perturbative rapidity anomalous dimension is universal and can be used to describe different data

Semi-inclusive deep inelastic scattering [Scimemi,AV,work in progress]

HERMES, COMPASS

$\chi^2/(N_p = 530) = 1.01$



Extracted non-perturbative rapidity anomalous dimension
is universal and can be used to describe different data

`artemide`

Program package for TMD phenomenology

- Efficient code based on TMD factorization
- Variety of evolution schemes (CSS, ζ-prescription, improved-$\mathcal{D}$, resummed, etc)
- Full control on non-perturbative and model inputs.
- All possible combinations of perturbative inputs LO,NLO,NNLO,“NNNLL”
- Bin-integrations, lepton cuts, etc.
- The library of processes is constantly updating
  - Drell-Yan -like
    - (unpolarized) Z/γ^*, W's, Higgs, pion-induces.
  - SIDIS
    - unpolarized
    - Sivers effect (in preparation)
  - More in plans

repository:
https://github.com/VladimirovAlexey/artemide-public

Part IV:
A bit on interpretation

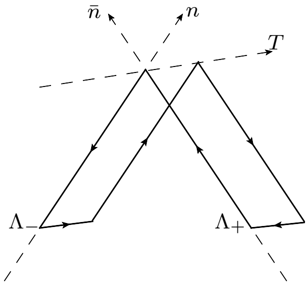
(in progress)

Rapidity anomalous dimension is function that directly measures properties of QCD vacuum
Which properties?

Non-perturbative definition of RAD

Rapidity anomalous dimension is independent on regularization.

$$S(b;\Lambda_+\Lambda_-)=\frac{\text{Tr}}{N_c}\langle 0|P\exp\left(-ig\int_C dx^\mu A_\mu(x)\right)|0\rangle$$



At $\Lambda_+\Lambda_-\to\infty$

$$S(b;\Lambda_+\Lambda_-)=\exp\left(-2\mathcal{D}(b)\ln(\Lambda_+\Lambda_-)+....\right)$$

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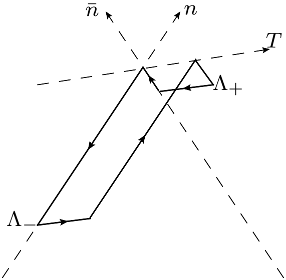
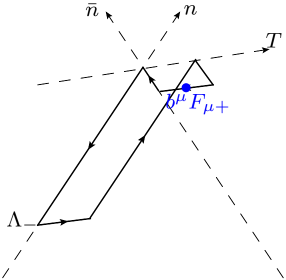


At $\Lambda_+\Lambda_-\to\infty$

$$S(b;\Lambda_+\Lambda_-)=\exp\left(-2\mathcal{D}(b)\ln(\Lambda_+\Lambda_-)+....\right)$$

Due to Lorentz invariance
enough $\Lambda_\pm\to\infty$

Rapidity anomalous dimension can be defined as a primary object.



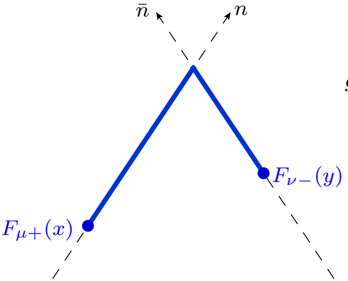
$$\mathcal{D}(b) = \frac{1}{2} \lim_{\Lambda_\pm \to \infty} \frac{S'(b; \Lambda_+ \Lambda_-)}{S(b; \Lambda_+ \Lambda_-)}$$

$$S'(b) = ig \int_0^1 d\beta \frac{\mathrm{Tr}}{N_c} \langle 0 | F_{b+}(-\Lambda_+ b + \beta b) P \exp\left(-ig \int_{C'} dx^\mu A_\mu(x) \right) |0\rangle$$

- Route to non-perturbative calculation and modeling.

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + ...$$

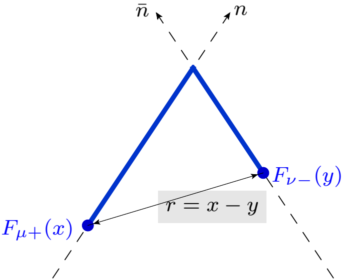


$\mathcal{D}_1$ is expressed via 2-point correlators connected by "a minimal distance link"

$$g^2\frac{\text{Tr}}{N_c}\langle 0|F_{\mu x}(x)[x,0][0,y]F_{\nu y}(y)|0\rangle =$$
$$\left(g^{\mu\nu} - \frac{y^\mu x^\nu}{(xy)}\right)\varphi_1(x,y) + (...)^{\mu\nu}\varphi_2$$

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + \ldots$$



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$$\left(g^{\mu\nu} - \frac{y^\mu x^\nu}{(xy)}\right)\varphi_1(x,y) + (\ldots)^{\mu\nu}\varphi_2$$

At LO $x^2 = y^2 = 0$
$\varphi_1(x,y) = \varphi_1(r^2)$
$\varphi_2(x,y) = 0$

$$\mathcal{D}_1(\mathbf{b}) = \frac{1}{2}\int_0^\infty \frac{d\mathbf{r}^2}{\mathbf{r}^2}\varphi_1(\mathbf{r}^2), \qquad \text{here } \mathbf{r}^2 = -r^2 > 0.$$

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + ...$$

Estimation

$$\mathcal{D}_1(\mathbf{b}) = \frac{1}{2}\int_0^\infty \frac{d\mathbf{r}^2}{\mathbf{r}^2}\varphi_1(\mathbf{r}^2), \qquad \text{here } \mathbf{r}^2 = -r^2 > 0.$$

- At $\mathbf{r}^2 \to 0$, $\varphi_1 \sim \mathbf{r}^2 \frac{\pi^2}{36} G_2$
- At $\mathbf{r}^2 \to \infty$, $\varphi_1 \sim \frac{1}{\mathbf{r}^2}$ (at least)

So, order-of-magnitude estimation

$$\mathcal{D}_1 \lesssim \frac{\pi^2}{72}\frac{G_2}{\Lambda^2_{\text{QCD}}} \sim (0.01 - 0.05)\text{GeV}^2$$

Extracted value

$$\mathcal{D}_1 = 0.022 \pm 0.009\text{GeV}^2$$

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$\mathcal{D}$ can be evaluated with non-Abelian Stockes theorem as an expansion

$$\mathcal{D}(b) = K_2 \otimes \langle FF \rangle + K_3 \otimes \langle FFF \rangle + ...$$

LO term

$$K_2 \otimes \langle FF \rangle = \frac{\mathbf{b}^2}{2} \int_0^\infty d\mathbf{r}^2 \int_0^1 d\alpha d\beta \frac{1}{\mathbf{r}^2 + 2(\xi-\alpha)(\beta-\xi)\mathbf{b}^2} \Big[\varphi_1 + \frac{2(\xi-\alpha)(\beta-\xi)\mathbf{b}^2}{\mathbf{r}^2 + 4(\xi-\alpha)(\beta-\xi)\mathbf{b}^2}\varphi_2\Big].$$

with $\varphi(\mathbf{r}^2 + (\alpha-\beta)^2\mathbf{b}^2, (\alpha-\xi)^2\mathbf{b}^2, (\beta-\xi)^2\mathbf{b}^2)$.

- Can be computed pertrubatively at small$-b$ (agrees!)
- Allows to study large-b asymptotic ($\to \sqrt{b^2}$).
- Allows model-interpretation.

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with $\varphi(\mathbf{r}^2 + (\alpha-\beta)^2\mathbf{b}^2, (\alpha-\xi)^2\mathbf{b}^2, (\beta-\xi)^2\mathbf{b}^2)$.

- Can be computed pertrubatively at small$-b$ (agrees!)
- Allows to study large-b asymptotic $(\to \sqrt{b^2})$.
- Allows model-interpretation. E.g. in SVM
  - $\varphi_i((x-y)^2, x^2, y^2) \to \varphi_i((x-y)^2)$
  - Especially simple formula
  - Static potential

$$V(\mathbf{b}) \quad = \quad \mathbf{b}\frac{\pi}{4}\mathcal{D}''(0) + \frac{\mathcal{D}'(0)}{2} + \frac{\mathbf{b}^2}{2}\int_{\mathbf{b}}^\infty d\mathbf{x} \frac{\mathcal{D}'(\mathbf{x})}{\mathbf{x}^2\sqrt{\mathbf{x}^2 - \mathbf{b}^2}}.$$

  - "String tension" $\sigma = \frac{\pi}{4}\mathcal{D}''(0) = \frac{\pi}{2}c_0 \simeq 0.03 \pm 0.01\text{GeV}^2$ vs. 0.19GeV^2

Conclusion

Nowadays,

- transverse momentum dependent (TMD) factorization theorem **is proved**,
- divergences of (TMD) soft factors **are understood**,
- rapidity anomalous dimension **are known at NNLO**
- TMD evolution **works!**.

Future

- global phenomenology DY+SIDIS, asymmetries
- interpretation and models,
- lattice measurements of $\mathcal{D}$,
- ...