Parton distributions with lattice QCD: **how it works.**

Alexey Vladimirov
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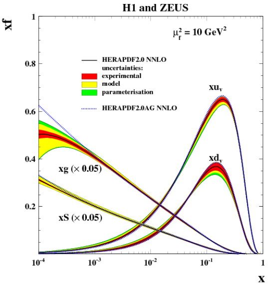
Intrinsic hadron structure is (probably) the most complicated topic in the QFT.
It is essentially **non-perturbative** (confinement!)

There are several method to study hadrons insides,
parton distributions is only one of them, but it is the one which is most well-known and well-explored



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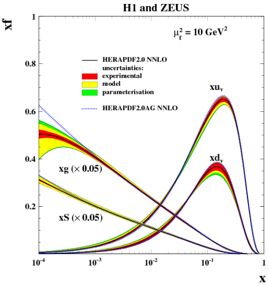
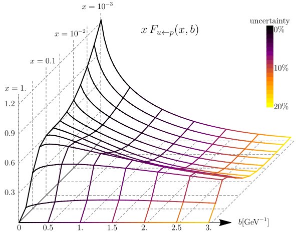
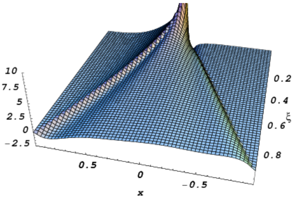
»»» PDFs, FFs, DAs, TMDs, GPDs, jetF, ... «««







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All these objects are **artificial**, but they could be obtained from the data by means of **factorization theorems**.

Factorization theorem is a statement of the form:

Observable$=\Big(\text{perturbative}\Big)\times\Big(\text{non-perturbative}\Big)+$ suppressed terms

[Drell, Yan,70]

$$\frac{d\sigma}{dQ^2}=\left(\frac{4\pi\alpha^2}{3Q^2}\right)\left(\frac{1}{Q^2}\right)\mathfrak{F}(\tau)=\left(\frac{4\pi\alpha^2}{3Q^2}\right)\left(\frac{1}{Q^2}\right)\int_0^1 dx_1\int_0^1 dx_2\,\delta(x_1x_2-\tau)\sum_a\lambda_a{}^{-2}F_{2a}(x_1)F_{2\bar{a}}{}'(x_2),$$

$\downarrow\downarrow\downarrow\downarrow\downarrow$

[today's arxiv]

$$\frac{d\sigma^c}{dy}=\sigma_{\mathrm{B}}^c(\mu_R^2)\sum_{a,b=q,\bar{q},g}\int_{x_1^0}^1\frac{dz_1}{z_1}\int_{x_2^0}^1\frac{dz_2}{z_2}\,f_a\left(\frac{x_1^0}{z_1},\mu_F^2\right)$$
$$\times f_b\left(\frac{x_2^0}{z_2},\mu_F^2\right)\Delta_{d,ab}^c(z_1,z_2,q^2,\mu_F^2,\mu_R^2).\qquad(1)$$

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$$\textbf{Observable} = \Big(\text{perturbative}\Big) \times \Big(\text{non-perturbative}\Big) + \text{suppressed terms}$$

Collinear factorization

- Theorem!
- Describes many "1D" observables
- Structure is understood at all levels
- PDFs, DAs, GPDs, FFs

TMD(-like) factorization

- Almost theorem
- Describes (some) 2D observables
- There are holes...
- TMDPDFs, TMDFFs

Naive factorization

- Model assumption
- Applied everywhere
- Often works very-well
- Jets, MC-generators, ...

I will try to explain basics of collinear and TMD factorization ⇒ lattice observables

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Part-1: collinear parton distributions on lattice

Note on terminology

Collinear factorization is about asymptotic behavior of operators.
The main tool is Operator Product Expansion (OPE)
[Wilson,Zimmermann,71]

$$A_1(x+\xi_1)\dots A_a(x+\xi_a) = \sum_{k=1}^{\infty} f_k(\xi_1,\dots,\xi_a)\,B_k(x)\,.$$

but in a much wider sense
The operators on LHS are generally **non-local**

Collinear factorization $\longleftrightarrow$ light-cone OPE

Collinear factorization is a statement about operator

$$O(z) = \sum_n (z^2)^n \sum_k C_{kn}(z) O_{kn}(z)$$

Some 1D operator
DIS$\to$ $J^\mu(z)J^\nu(0)$

Coefficient functions

Full set of
"lower-dimension" operators

$$J^\mu(z)J^\nu(0) = \frac{C_0}{z^6} + C_1 \frac{\bar{q}(z)\ \not{z} q(0)}{z^4} + ...$$

Collinear factorization $\longleftrightarrow$ light-cone OPE

Collinear factorization is a statement about operator

$$\underbrace{Z_o O(z,\mu)}\ =\ \sum_n (z^2)^n \sum_k \underbrace{Z_o C_{kn}(z,\mu,\mu_F) Z_C^{-1}}_{\mathbf{C}_{nk}(z,\mu,\mu_F)} \underbrace{Z_C O_{kn}(z,\mu_F)}$$

- Divergences of cancel between coefficient functions and operators
- Physically interesting to consider not a local limit (Wilson-Zimmermann OPE) $z^\mu \to 0$, but $z^2 \to 0$
- Gives rise to the notion of twist

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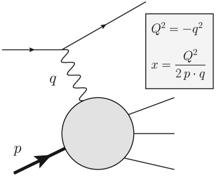
Difference between local and light-cone expansion

DIS : $W^{\mu\nu}(q,p) \sim \int d^4 z e^{iqz} \langle p | J^\mu(z) J^\nu(0) | p \rangle$



Bjorken limit: $Q^2 \gg p^2, \qquad (pq) \sim Q^2 \;(x = \text{fixed})$

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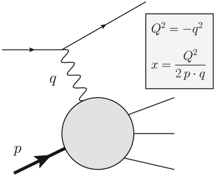

Bjorken limit: $Q^2 \gg p^2$, $(pq) \sim Q^2$ (x = fixed)

1. Expanding operator: $\int d^4z e^{iqz} \langle p| \sum z^{\mu_1}...z^{\mu_n} \; C_n \otimes O_{\mu_1...\mu_n}(0)|p\rangle$

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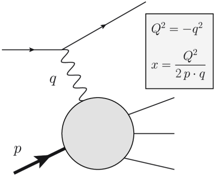

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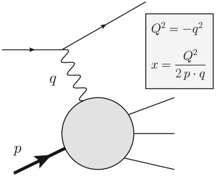

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3. Evaluating: $\left(\frac{q^{\mu_1}...q^{\mu_n}}{[Q^2]^{n+2}} + .. + \frac{q^{\mu_1}..g^{\mu_i\mu_j}..q^{\mu_n}}{[Q^2]^{n+1}} + ...\right) C_n \otimes f_n \quad p_{\mu_1}...p_{\mu_n}$

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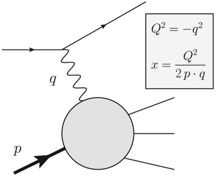

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4. Contracting: $\frac{1}{Q^4}\left(x^{-n} + .. + x^{-n-2}\frac{p^2}{Q^2} + ...\right) C_n \otimes f_n$

The leading term is given by symmetric traceless part of the operator

In DIS the leading term is given by a collection of terms
These terms are characterized by a quantum number "twist"
twist=dimension-spin

$$\bar{q}\gamma^\mu\partial^\nu q = \underbrace{\bar{q}\left(\frac{\gamma^\mu\partial^\nu+\gamma^\nu\partial^\mu}{2} - g^{\mu\nu}\ \not{\partial}\right)q}_{\text{tw-2}} + \underbrace{\bar{q}\left(\frac{\gamma^\mu\partial^\nu-\gamma^\nu\partial^\mu}{2}\right)q}_{\text{tw-3}} + g^{\mu\nu}\underbrace{\bar{q}\ \not{\partial}q}_{\text{tw-4}} \tag{1}$$

Each operator can be decomposed into components with definite twist

$$O^{\mu_1...\mu_n} = \sum_{\text{tw}} A^{\mu_1...\mu_n}_{\text{tw};\nu_1...\nu_n} O^{\nu_1...\nu_n}_{\text{tw}}$$

In fact, it is a decomposition over irreducible representations of Lorentz-group

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These terms are characterized by a quantum number "twist"
twist=dimension-spin

$$W^{\mu\nu} \sim \int d^4 z e^{iqz} \sum_n z^{\mu_1}...z^{\mu_n} C_n \otimes \langle p|O_{\mu_1...\mu_n}(0)|p\rangle$$
$$\sim \int d^4 z e^{iqz} \sum_n z^{\mu_1}...z^{\mu_n} \sum_{\text{tw}=2}^{\infty} C_n^{\text{tw}} \otimes \langle p|O^{\text{tw}}_{\mu_1...\mu_n}(0)|p\rangle$$
$$\sim \frac{1}{Q^4}\sum_n \left(x^n C_n^{\text{tw-2}} f_n^{\text{tw-2}} + \frac{1}{Q^2} x^{n-2} C_n^{\text{tw-4}} f_n^{\text{tw-4}} + ...\right)$$

- Operators with different twist do not mix with each other (irreducible Lorentz representation)
  - Preserved by evolution!
- Twist $\neq$ power
  - Indeed, in DIS, at Q^{-n} appear only operators with twist $< n$
  - Possible kinematical factors
  - Equation of motions!

Back to non-local definition!

The sum over n and twist-decomposition commutes!

$$\sum_n z^{\mu_1}...z^{\mu_n} O^{\text{tw-n}}_{\mu_1...\mu_n} = z^{2[n/2]} O^{\text{tw-n}}(z)$$

- Also for quantum operators [Anikin,Zavyalov,78]

Light-cone OPE:

$$O(z) = \sum_n z^{2[n/2]} C_{\text{tw}-n} \otimes O^{\text{tw-n}}(z)$$

Here, operator does not have z^2-contribution $\Rightarrow$ light-cone

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Light-cone without light-cone:

it is essential to realize that (the function is independent on z^2) $\neq f(z^2 = 0)$

Famous and important example:

$$\bar{q}(z)\gamma^\mu q(0) = \underbrace{\int_0^1 du \frac{\partial}{\partial z_\mu}\bar{q}(uz)\not{z}q(0)}_{\text{tw-2}} + \underbrace{\int_0^1 du \int_0^u dv z^\nu \bar{q}(uz)\not{z}F_{\mu\nu}(vz)q(0) + ...}_{\text{tw-3}} + z^2...$$

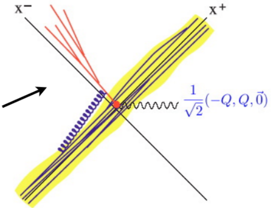
Hint: integrate by parts, and expand at $z^\mu \to 0$.

$$\bar{q}(z)\not{z}q(0) = \text{twist-2} + z^2\text{twist-4}$$

Common terminology: light-cone decomposition

$$v^\mu = n^\mu v^- + \bar{n}^\mu v^+ + v_T^\mu$$

$$n^2 = \bar{n}^2 = 0, \qquad (n\bar{n}) = 1, \qquad (nv_T) = (\bar{n}v_T) = 0.$$



- $$n^\mu = \{1, 0, 0, 1\}/\sqrt{2}$$
  $$\bar{n}^\mu = \{1, 0, 0, -1\}/\sqrt{2}$$
- Hadron momentum ($p^2 = m^2$)
  $$p^\mu = \bar{n}^\mu p^+ + n^\mu \frac{m^2}{2p^+}$$
  $$p^+ \gg p^-$$

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Some examples

$$\text{twist-2} \rightarrow \quad \bar{q}(nz^+)\gamma^+ q(0) \tag{1}$$

$$\text{twist-3} \rightarrow \quad \bar{q}(nz^+)q(0) \sim \int \bar{q}\sigma^{\mu+}F_{\mu+}q(0) \tag{2}$$

$$\text{indefinite} \rightarrow \quad \bar{q}(nz^+)\gamma^\mu q(0) \sim \bar{n}^\mu \underbrace{\bar{q}(nz^+)\gamma^+ q(0)}_{\text{tw-2}} + \underbrace{\bar{q}(nz^+)\gamma_T^\mu q(0)}_{\bar{q}\gamma^+ F_{\mu+} q = \text{tw-3}} + n^\mu \underbrace{\bar{q}(nz^+)\gamma^- q(0)}_{\text{tw-4}} \tag{3}$$

$$\text{tw-3} \rightarrow \quad \bar{q}(nz^+)\gamma^+ \partial_{\mu_T} q(0) \tag{4}$$

$$\text{tot.der.tw-2} \rightarrow \quad \partial_\mu \{\underline{\bar{q}(nz^+)\gamma^+ q(0)}_{\text{tw-2}}\} \tag{5}$$

Don't be confused by modern terminology!

- Geometrical twist $\equiv$ "dimension - spin "
- Collinear twist $\sim$ "dimension"
- Dynamical twist $\sim$ power of Q^2 in structure function
- Conformal twist
- TMD twist
- ...

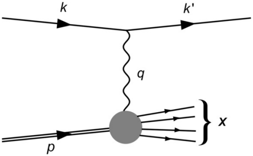
Same OPE → different observables

OPE is unique expression

$$J^{\mu}(x)J^{\nu}(0) = -\frac{x^{\mu}g^{\nu\alpha} + x^{\nu}g^{\mu\alpha} - x^{\alpha}g^{\mu\nu}}{(4\pi)^2 x^4}\int dy K(x,y,\alpha_s)[\bar{q}(y)\gamma_{\alpha}q(0)] + ...$$

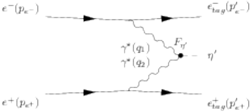
Deep inelastic scattering (DIS) $e + p \to e + X$

$$d\sigma \sim L_{\mu\nu}W^{\mu\nu} = L_{\mu\nu}\int e^{i(qz)}\langle p|J^{\mu}(z)J^{\nu}(0)|p\rangle$$



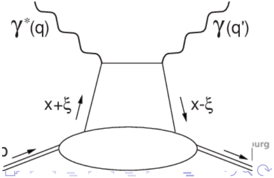
Meson production $\gamma^* + \gamma^* \to \eta$

$$A = e_{\mu}e_{\nu}\int e^{i((q_1+q_2)z)}\langle 0|J^{\mu}(z)J^{\nu}(0)|p\rangle$$



Deeply virtual Compton Scattering (DVCS) $\gamma^* + p \to \gamma + p$

$$A = e_{\mu}e_{\nu}\int e^{i((q_1+q_2)z)}\langle p'|J^{\mu}(z)J^{\nu}(0)|p\rangle$$



Single operator → different matrix elements → different observables

$$O(z)=\bar q(z)\not z q(0)|_{\text{tw-2}}$$

Parton distribution function (PDF)

$$\langle p|O(z)|p\rangle \quad=\quad 2(pz)\int dx e^{ix(pz)} f_1(x) \tag{6}$$

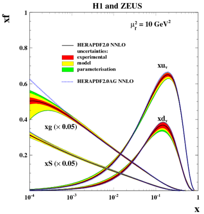
Distribution amplitude (DA)

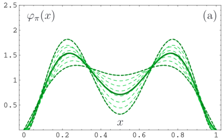
$$\langle 0|O(z)|p\rangle \quad=\quad 2(pz)\int dx e^{ix(pz)} \varphi(x) \tag{7}$$

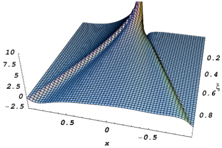
Generalized parton distribution

$$\langle p|O(z)|p'\rangle \quad=\quad 2(pz)\int dx d\xi e^{i(x+\xi)(pz)+i(x-\xi)(pz)} H(x,\xi) \tag{8}$$

+ Generalized distribution amplitudes







How to measure parton distributions on lattice?

In principle...

- If we would be able to study the hadron-to-hadron expectation value of any reasonable non-local operator at $z^2 \to 0$ in that would give us information about PDFs.
- It does not matter which direction is z. It also could be $z^\mu = \hat{n}_3^\mu z$
- Realized long ago
  [V.Braun, P.Gornicki, L.Mankiewicz, 94], [P.Hoyer,M.Vanttinen,96] ...
  [V.Braun,D.Muller,06]

Ji's paper

- Xiangdong Ji, Phys.Rev.Lett. 110 (2013) $\to$ explosion of works

Simplest operators is PDF operator it-self!

$$O^\mu(z) = \bar{q}(z)[z,0]\gamma^\mu q(0)$$

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Single operator $\rightarrow$ Single matrix element $\rightarrow$ Different observables (**What?!**)

Ioffe-time distribution (ITD) [V.Braun,at al, 9410318]

$$\langle p|\bar{q}(z)\gamma^{\mu}[z,0]q(0)|p\rangle = 2p^{\mu}\mathcal{M}((pz),z^2) + 2z^{\mu}\mathcal{N}((pz),z^2) \tag{9}$$

- That is what measured on the lattice .
- $\mathcal{M} \sim (\text{twist}-2) + z^2...$
- $\mathcal{N} \sim (\text{twist}-2/4) + z^2...$

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$$f_1(x) = \int \frac{d(pz)}{2\pi} e^{-ix(pz)}\mathcal{M}((pz),z^2=0) \tag{10}$$

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Pseudo-parton distribution function pseudoPDF [A.Radyuskin,17]

$$P(x,z^2) = |z| \int_{-\infty}^{\infty} \frac{d(pv)}{2\pi} e^{-ix|z|(pv)} \mathcal{M}((pz), z^2) \tag{11}$$

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Quasi-parton distribution function quasiPDF [X.Ji,et al,14]

$$\tilde{Q}(x,|p|^2) = |p| \int_{-\infty}^{\infty} \frac{d(vz)}{2\pi} e^{-ix|p|(vz)}\mathcal{M}((pz),z^2) \tag{12}$$

Theory-wise quasiPDF and pseudoPDF are same.
But practically there is a lot of difference

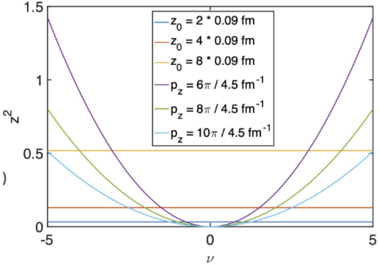
$$P(x,z^2) = |z| \int_{-\infty}^{\infty} \frac{d(pv)}{2\pi} e^{-ix|z|(pv)} \mathcal{M}((pz),z^2)$$

$$\tilde{Q}(x,|p|^2) = |p| \int_{-\infty}^{\infty} \frac{d(vz)}{2\pi} e^{-ix|p|(vz)} \mathcal{M}((pz),z^2)$$

The problem is power corrections

- If $p_z \to \infty$ both pictures coincides
- pseudo-PDF preserves structure of "original" power expansion
- quasi-PDF mixes terms "original" power expansion

[picture by J.Karpie]



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Role of the large momentum

$$\bar{q}(z)\gamma^\mu[z,0]q(0) = \int_0^1 du \frac{\partial}{\partial z^\mu} \underbrace{\bar{q}(uz)\ \not{z}[uz,0]q(0)}_{\text{tw-2}} (1+\alpha_s ...) + z^2[\text{tw-3/4}] + ...$$

$$\langle p|\bar{q}(z)\gamma^\mu[z,0]q(0)|p\rangle = 2p^\mu \int_{-1}^1 dx e^{ix(pz)} \underbrace{f_1(x, x(pv))}_{\text{PDF}} (1+\alpha_s ...) + z^2[\text{tw-2/3/4}] + ...$$

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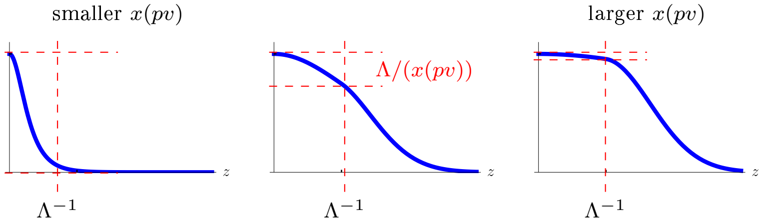
$$\bar{q}(z)\gamma^\mu[z,0]q(0) = \int_0^1 du \frac{\partial}{\partial z^\mu} \underbrace{\bar{q}(uz)\ \not{z}[uz,0]q(0)}_{\text{tw-2}} (1+\alpha_s...) + z^2[\text{tw-3/4}] + ...$$

$$\langle p|\bar{q}(z)\gamma^\mu[z,0]q(0)|p\rangle = 2p^\mu \int_{-1}^1 dx e^{ix(pz)} \underbrace{f_1(x, x(pv))}_{\text{PDF}} (1+\alpha_s...) + z^2[\text{tw-2/3/4}] + ...$$



Larger (pv) = smoother asymptotic $z^2 \to 0$.

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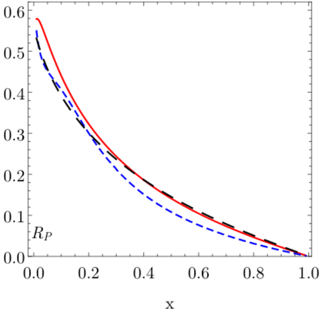
How important power corrections?

[V.Braun,AV,J.H.Zhang,1810.00048]

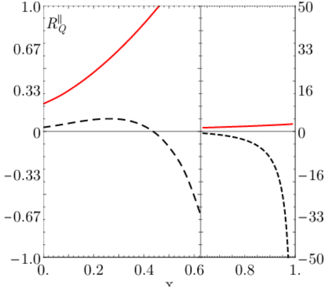
PseudoPDF

$$P(x,z^2)=f(x)\left(1+v^2z^2\Lambda^2R_P(x)\right)$$



QuasiPDF

$$Q(x,(pv)^2)=f(x)\left(1-\frac{v^2\Lambda^2}{x^2\bar{x}(pv)^2}R_Q(x)\right)$$



Life is not so simple..

Topics that I left

- Perturbative computations (some already computed at NNLO!).
- Renomalization (linear divergences)
- Normalization (infinitely many ways, some cases effectively clean-out power corrections)
- Higher twist observables (very new and hot topic)
- Correlation of different operators (often much simple to measure on lattice. Smaller noise!)
- ...

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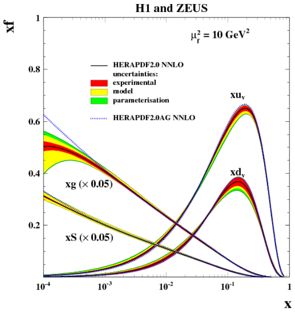
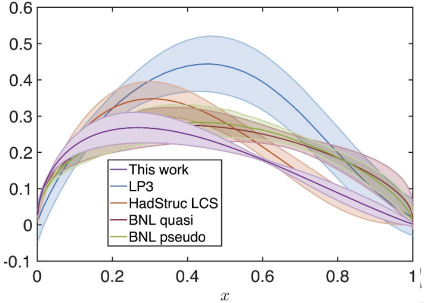
Why measurement of PDFs on lattice is interesting?

In foreseen future lattice cannot compete with experiment in PDF-measurements

- Maximum $P \sim 2-3$GeV
- Only connected contributions (non-singlet PDF), no gluons..
- Almost unknown systematics

[picture by J.Karpie]





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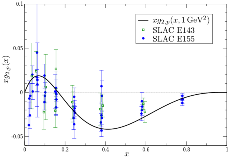
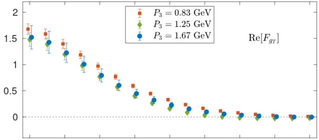
- Almost unknown systematics

But lattice has access to physically unmeasurable PDFs

- Higher twist PDFs
- "Exotic states" (kaon PDFs)
- "Exotic regimes" $H(x,x)$
- etc.

[picture by K.Cichy]





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Part-2:
transverse momentum dependent (TMD) distributions on lattice

TMD factorization $\neq$ collinear factorization

- It could not be related to local OPE — however [V.Moos,AV;2008.01744]
- It deals with different type of divergences
- More degrees of freedom

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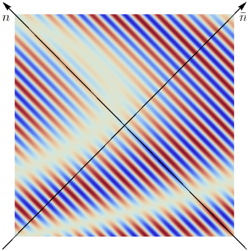
Light-cone OPE via background field method

e.g.[Abbott,81;Braun,Balitsky,89,..]

$$\langle p|O|p\rangle = \mathcal{N}^{-1}\int D\bar q DqDA\; e^{iS_{QCD}}\;\bar\Psi[\bar q,q,A]\;O[\bar q,q,A]\;\Psi[\bar q,q,A]$$

Let me assume that accelerated hadron is made of the fields that move collinearly



$$\Psi[\bar q,q,A]\to\Psi[\bar q_c,q_c,A_c],\qquad \partial_\mu q_c\sim\{1,\lambda^2,\lambda\}$$

$$q=q_c+q_h,\qquad A=A_c+A_h$$

$$Dq\to Dq_cDq_h,\qquad DA=DA_cDA_h$$

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e.g.[Abbott,81;Braun,Balitsky,89,..]

$$\langle p|O|p\rangle = \mathcal{N}^{-1}\int D\bar{q}DqDA\; e^{iS_{QCD}}\;\bar{\Psi}[\bar{q},q,A]\;O[\bar{q},q,A]\;\Psi[\bar{q},q,A]$$

Integrate over fast components

$$\langle p|O|p\rangle = \mathcal{N}^{-1}\int D\bar{q}DqDA\; e^{iS_{QCD}}\;\bar{\Psi}[\bar{q},q,A]\;O^{eff}[\bar{q},q,A]\;\Psi[\bar{q},q,A]$$

$$\begin{aligned} O^{eff}[\bar{q},q,A] &= \mathcal{N}^{-1}\int D\bar{q}_h Dq_h DA_h\; e^{iS^h_{QCD}+iS^{back}_{QCD}}\;O^{eff}[\bar{q}+\bar{q}_h, q+q_h, A+A_h] \\ &= \underbrace{\sum_n C_n\otimes O_n[\bar{q},q,A]}_{\text{light-cone OPE}} \qquad\qquad O_n\sim\lambda^n \end{aligned}$$

The simplest way to deal with OPE for power suppressed terms

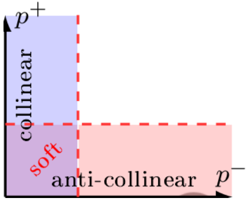
In TMD factorization one considers two oppositely moving hadrons
$p + p \to \gamma + X$

$$\langle p_1, p_2|O|p_1, p_2\rangle = \mathcal{N}^{-1} \int D\bar{q}DqDA\ e^{iS_{QCD}}\ \bar{\Psi}_1\ \bar{\Psi}_2\ \ O[\bar{q}, q, A]\ \ \Psi_1\ \Psi_2$$

Each hadron is made of the fields that move collinearly

$$\Psi_1[\bar{q}, q, A] \to \Psi_1[\bar{q}_c, q_c, A_c], \qquad \partial_\mu q_c \sim \{1, \lambda^2, \lambda\}$$

$$\Psi_2[\bar{q}, q, A] \to \Psi_2[\bar{q}_{\bar{c}}, q_{\bar{c}}, A_{\bar{c}}], \qquad \partial_\mu q_{\bar{c}} \sim \{\lambda^2, 1, \lambda\}$$



Double counting in the "soft" region $\{\lambda^2, \lambda^2, \lambda\}$

To solve this issue → introduce extra "soft" modes and subtract them (complicated story)

In TMD factorization in background formulation

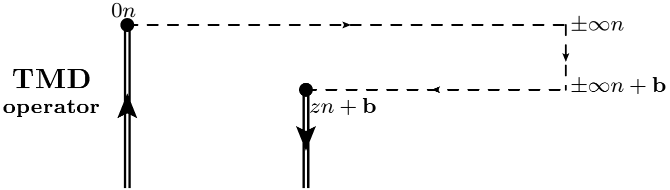
[I.Balitsky,18;AV,in prep.]

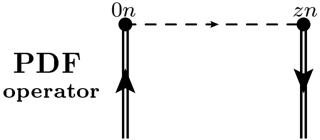
disclaimer: no strict proof, but it works for known cases

$$\langle p_1, p_2|O|p_1, p_2\rangle = \int [D\bar{q}DqDA]\ e^{iS_{QCD}}\ \bar{\Psi}_1 O_1^{\text{eff}} \Psi_1 \times \mathcal{S} \times \int [D\bar{q}DqDA]\ e^{iS_{QCD}}\ \bar{\Psi}_2 O_2^{\text{eff}} \Psi_2$$

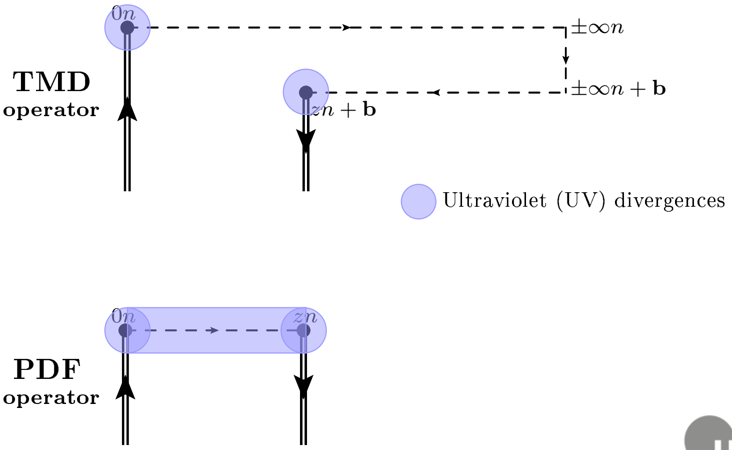
$$O_1^{\text{eff}} \mathcal{S} O_2^{\text{eff}} = \frac{\int [D\bar{q}DqDA] e^{iS_{QCD}+iS_{QCD}^{back}} O[q_h + q_c + q_{\bar{c}} + q_s, ..]}{\left(\int [D\bar{q}DqDA] e^{iS_{QCD}+iS_{QCD}^{back}} O[q_h + q_s, ..]\right)^2}$$

TMD operator vs. PDF operator
"New" type of divergences → rapidity divergences



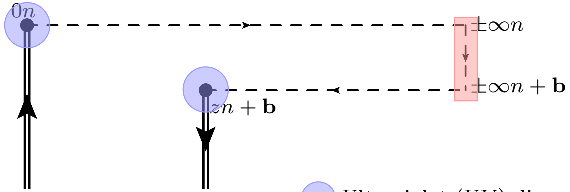


TMD operator vs. PDF operator
"New" type of divergences → rapidity divergences



TMD operator vs. PDF operator
“New” type of divergences → rapidity divergences

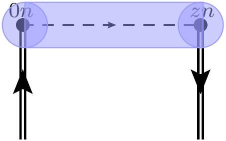
TMD operator



Ultraviolet (UV) divergences

Rapidity divergences

PDF operator



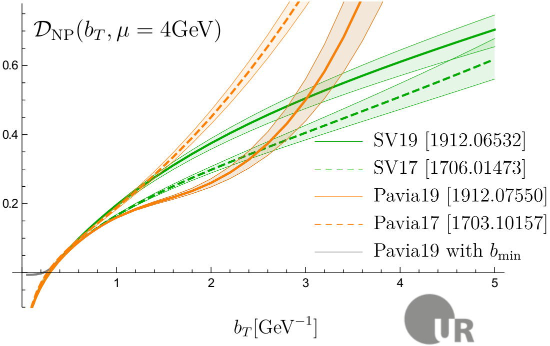
Rapidity divergences
are multiplicatively renormalizable
[AV,JHEP 1804 (2018)]
But the renormalization factor
contains non-perturbative element

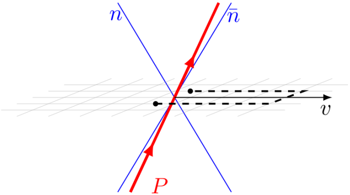
Collins-Soper kernel

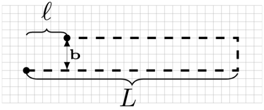
$$\frac{dO_{\mathrm{TMD}}(b;\mu,\zeta)}{d\ln\mu} = \gamma_F(\mu,\zeta)O_{\mathrm{TMD}}(b;\mu,\zeta) \qquad \frac{dO_{\mathrm{TMD}}(b;\mu,\zeta)}{d\ln\zeta} = -\mathcal{D}(b,\mu)O_{\mathrm{TMD}}(b;\mu,\zeta)$$

- CS-kernel dictates evolution for TMD distribution
- Is a non-perturbative function
- Perturbative at small-b (known to NNLO=three-loops)
- Describes QCD vacuum properties [AV;2003.02288]
- Extracted from data

Phenomenological extractions







Restrictions on observable

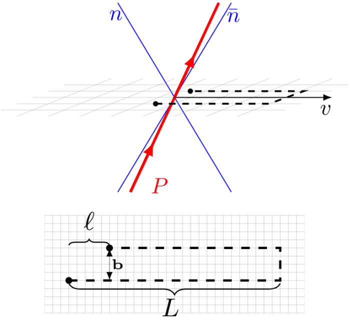
- Equal-time
- With transverse size $(bP) = 0$
- With anti-collinear modes

Simplest case:

$$W_{f\leftarrow h}^{[\Gamma]}(b;\ell,L;v,P,S) = \frac{1}{2}\langle P,S|\bar{q}_f(b+\ell v)\Gamma \times [b+\ell v, b+Lv][b+Lv, Lv][Lv,0]q_f(0)|P,S\rangle, \quad (1)$$

Γ = some Dirac structure

Constructing TMD-sensitive observable suitable for lattice



Restrictions on observable

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$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S) = \frac{1}{2}\langle P,S|\bar{q}_f(b+\ell v)\Gamma \times [b+\ell v, b+Lv][b+Lv, Lv][Lv,0]q_f(0)|P,S\rangle, \qquad (1)$$

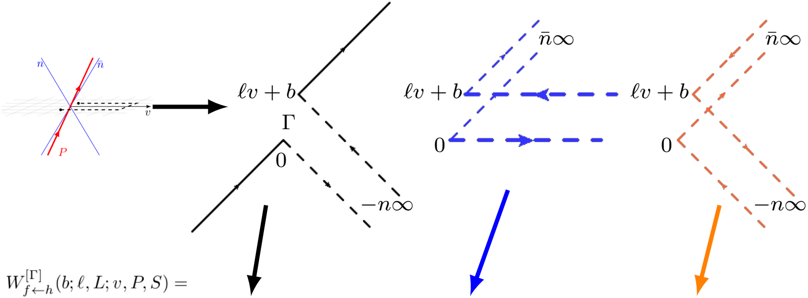
Γ = some Dirac structure

It is like DIS+(instant)jet

$$\text{At } L\to\infty \qquad [0,Lv]\to H(0) \qquad (\text{with } \mathcal{L}_{HH} = H^\dagger(ivD)H)$$

$$\text{current } J_i(x) = H^\dagger(x)q(x), \qquad \text{hadron tensor } W_{ij} = \langle P|J_i^\dagger(x)J_j(0)|P\rangle$$

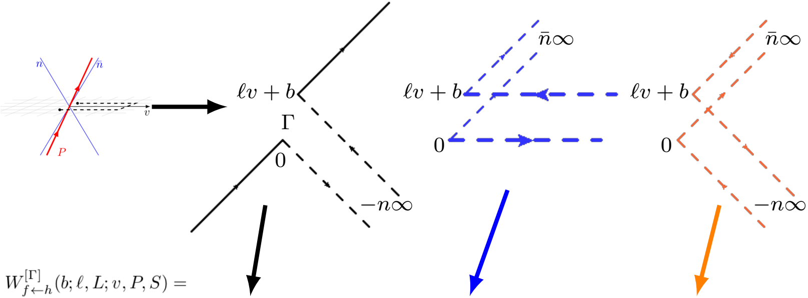
Factorization is (almost) equivalent to factorization of SIDIS or DY



$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S)=$$

$$\left|C_H(\hat{p}v)\right|^2\langle P,S|\left(\bar{\xi}W_n(b+\ell v)\frac{\Gamma}{2}W_n^\dagger\xi(0)\right)\left(H^\dagger W_{\bar{n}}(0)W_{\bar{n}}^\dagger H(b+\ell v)\right)\frac{\mathrm{Tr}}{N_c}\left[Y_n^\dagger Y_{\bar{n}}(b+\ell v)Y_{\bar{n}}^\dagger Y_n(0)\right]|P,S\rangle.$$

Factorization is (almost) equivalent to factorization of SIDIS or DY



$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S) =$$
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Negelecting power corrections and accounting the overlap in the soft modes

$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S) = \left|C_H(\hat{p}v)\right|^2 \widetilde{\Phi}^{[\Gamma']}_{f\leftarrow h}(b,\ell v^-;P,S)\widetilde{\Psi}(b,\ell v^+;v)\frac{S(b)}{\text{Z.b.}}$$

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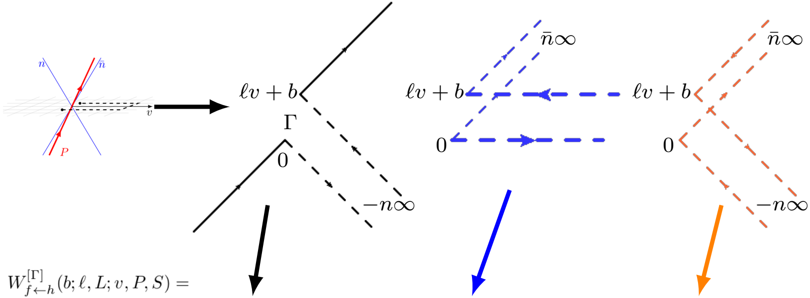
Factorization is (almost) equivalent to factorization of SIDIS or DY



$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S) =$$
$$\left|C_H(\hat{p}v)\right|^2 \langle P,S| \left(\bar{\xi}W_n(b+\ell v)\frac{\Gamma}{2}W_n^\dagger \xi(0)\right)\left(H^\dagger W_{\bar{n}}(0)W_{\bar{n}}^\dagger H(b+\ell v)\right)\frac{\mathrm{Tr}}{N_c}\left[Y_n^\dagger Y_{\bar{n}}(b+\ell v)Y_{\bar{n}}^\dagger Y_n(0)\right]|P,S\rangle.$$

Negelecting power corrections and accounting the overlap in the soft modes

$$W^{[\Gamma]}_{f\leftarrow h}(b;\ell,L;v,P,S) = \left|C_H(\hat{p}v)\right|^2 \widetilde{\Phi}^{[\Gamma']}_{f\leftarrow h}(b,\ell v^-;P,S)\widetilde{\Psi}(b,\ell v^+;v)\frac{S(b)}{\text{Z.b.}}$$

operator of parton's momentum ($\hat{p}\sim xP$)
Fourier conjugated to ℓ

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Sources and sizes of power corrections

$$W^{[\Gamma]} = |C_H\left(\frac{\hat{p}v}{\mu}\right)|^2\ \Phi^{[\Gamma']}(b,\ell v^-;\mu,\zeta;P,S)\ \ \Psi(b;\mu,\bar\zeta;v) \quad + \textbf{power corrections}$$

- $\frac{P^-}{x^2P^+}$ and $\frac{\ell}{L}$ from collinear/anti-collinear modes separation
- $\frac{1}{x|b|P^+}$ from collinear/transverse modes separation
- $\frac{b}{L}$ from anti-collinear/transverse modes separation
- $\ell\Lambda_{\text{QCD}}$ to remove ℓ-dependence from Ψ

Factorization limit: $L\to\infty$, $P^+\to\infty$, b-fixed(non-zero), ℓ-fixed(also zero).
No need for Fourier transform!

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$\Psi(b;\mu,\bar{\zeta};v)$ is generally unknown (see however X.Ji, Liu,Liu, 2019)
It cancels in the ratios with same b.
If the same v, ℓ L are taken the Wilson-line renormalization factor also cancel.

$$R = \frac{W^{[\Gamma_1]}_{f_1\leftarrow h_1}(b;\ell,L,v;P_1,S_1;\mu)}{W^{[\Gamma_2]}_{f_2\leftarrow h_2}(b;\ell,L,v;P_2,S_2;\mu)}$$

$$= \frac{|C_H\left(\frac{\hat{p}v}{\mu}\right)|^2\ \Phi^{[\Gamma_1']}_{f_1\leftarrow h_1}(b,\ell v^-;\mu,\zeta;P_1,S_1)}{|C_H\left(\frac{\hat{p}v}{\mu}\right)|^2\ \Phi^{[\Gamma_2']}_{f_2\leftarrow h_2}(b,\ell v^-;\mu,\zeta;P_2,S_2)} + \textbf{power corrections}$$

Plenty of information/tests

- Test power corrections! E.g. $\Gamma_1=\gamma^-$ and $\Gamma_2=\gamma^+$ then $R=$ **power corrections**
- Extract moments of TMDs
- ..
- Collins-Soper kernel

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Extraction of Collins-Soper kernel $\mathcal{D}$
Ratio at different $P_{1,2}$ and rest all the same

[Ebert,Stewart,Zhao,1811.00026] [Schäfer,AV,2002.07527]

To facilitate cancellation set $\ell = 0$

$$R_{P_1/P_2} = \left(\frac{P_2^+}{P_1^+}\right)^{2\mathcal{D}(b,\mu)+1} \frac{\int dx \left|C_H\left(\frac{xv^- P_1^+}{\mu}\right)\right|^2 \Phi^{[\Gamma']}_{f\leftarrow h}(x,b)|x|^{-2\mathcal{D}(b,\mu)}}{\int dx \left|C_H\left(\frac{xv^- P_2^+}{\mu}\right)\right|^2 \Phi^{[\Gamma']}_{f\leftarrow h}(x,b)|x|^{-2\mathcal{D}(b,\mu)}} + \textbf{power corr.}$$

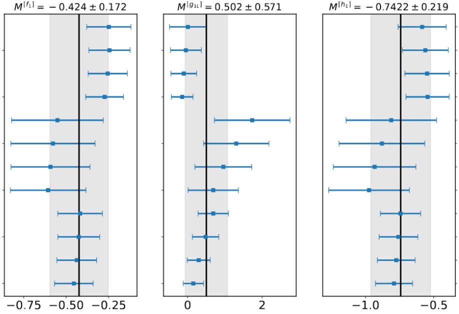
TMDs do not cancel only due to perturbative logarithms (here $\mu = 2|v^-|\sqrt{P_1^+ P_2^+}$)

$$R_{P_1/P_2}(\ell = 0) = \left(\frac{P_2^+}{P_1^+}\right)^{2\mathcal{D}(b,\mu)+1} \mathbf{r} + \textbf{power corrections}$$

$$\mathbf{r} = 1 + 4C_F \frac{\alpha_s(\mu)}{4\pi} \ln\left(\frac{P_1^+}{P_2^+}\right)\left[1 - 2M^{\Gamma}_{\ln|x|}(b,\mu)\right] + \mathcal{O}(\alpha_s^2)$$

$$M^{\Gamma}_{\ln|x|}(b,\mu)=\frac{\int dx \ln|x| |x|^{-2\mathcal{D}+1}\Phi^{[\Gamma]}(x,b)}{\int dx |x|^{-2\mathcal{D}+1}\Phi^{[\Gamma]}(x,b)}$$

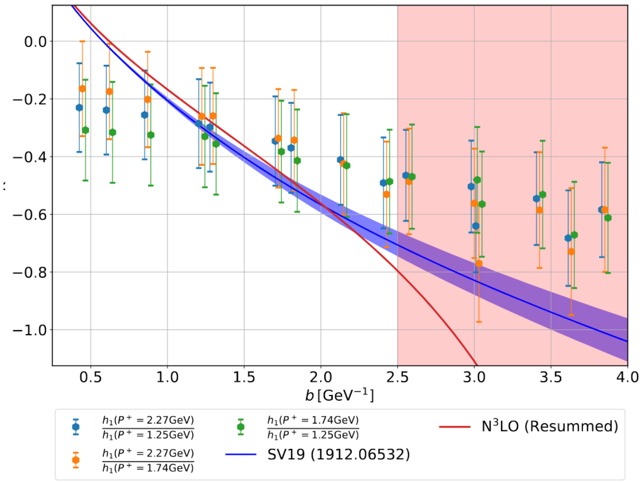
M constant (in b) lets determine it on the lattice



- Various convigurations
- Roughly (±50%) agrees with phenomenology
- Also contains part of power correction!

Collins-Soper kernel on lattice

PRELIMINARY RESULT



ERROR BARS INCLUDE SYSTEMATICS

Conclusion

(General)Theory background is absolutely solid

- Factorization theorem is the operator statement

Real life

- There are plenty of "technical" problems (small P, normalization, lattice noise, etc.)
- Not all "definitions" equally good (e.g. quasiPDF vs. pseudoPDF)

Nonetheless, I am sure that lattice can/must/will greatly contribute to the understanding of parton picture

although may be not in straightforward way