Evolution of transverse momentum dependent distributions

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Mainz

Introduction

The talk is a mini-review about

- transverse momentum dependent (TMD) factorization theorems,
- soft factors,
- rapidity divergences,
- rapidity anomalous dimension and TMD evolution,
- and its interpretation.

Plan of the talk

- introduction to TMD factorization,
- TMD soft factor and rapidity divergences,
- renormalization theorem for rapidity divergences,
- evolution equation and ζ-prescription
- comparison with the data
- interpretation of rapidity anomalous dimension

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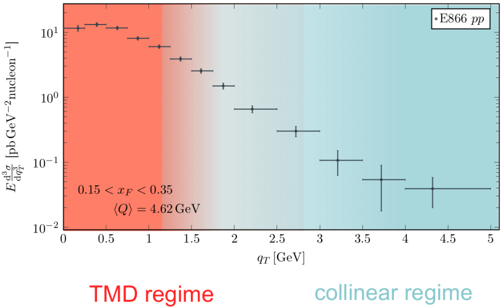
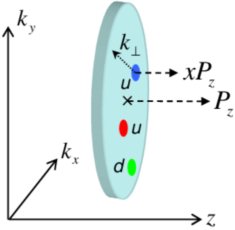
Transverse momentum dependent (TMD) factorization describes double-inclusive processes in the regime of small transverse momentum ($q_T^2 \ll Q^2$)

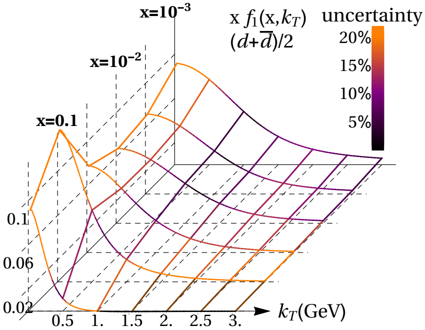
processes: $h_1 + h_2 \to \gamma^*/Z/W + X$ "Drell-Yan"
$h_1 + \gamma^* \to h_2 + X$ semi-inclusive DIS (SIDIS)
$e^+e^- \to h_1 + h_2 + X$



The transverse momentum of photon q_T is determined with respect to "hadron plane"

In TMD regime the produced transverse momentum is mostly of "non-perturbative" origin: $\Rightarrow$ TMD distributions (PDFs and FFs)





[Bertone,Scimemi,AV,1902.08474]

TMD distributions should not be mistaken with collinear distributions (although they have some common points).

- Structurally different: different divergences and different evolution.

Part I:
TMD factorization

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Structure of TMD factorization

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim L_{\mu\nu}\int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

Field-modes factorization

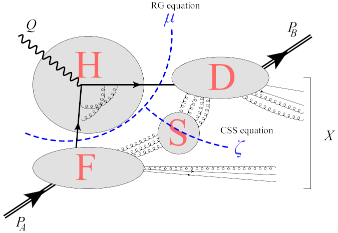
TMD soft factor

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\ \ \Phi_{h_1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T) + Y$$

power suppressed terms

TMD FF

TMD PDF



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Field-modes factorization

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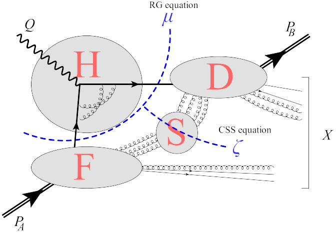
TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



All components of factorization formula contain rapidity divergences.

- TMD PDF: divergences in n-sector
- TMD FF: divergences in $\bar{n}$-sector
- TMD SF: divergences in n & $\bar{n}$-sector

Summation of soft gluon exchanges $\Rightarrow$ Wilson lines

$$[x, y] = P \exp\left(ig \int_x^y dz^\mu A_\mu(z)\right)$$

Parallel transporter of a gluon field.

- Sums soft-exchanges between hadron and parton
- Presented in all elements of factorization theorems

Example: parton distribution function

$$f(x) = \int \frac{d\lambda}{2\pi} e^{ixp\lambda} \langle \text{hadron} | \bar{q}(\lambda n)[\lambda n, 0] q(0) | \text{hadron} \rangle$$

$n^2 = 0$
on light-cone

Reminder

Summation of soft gluon exchanges $\Rightarrow$ Wilson lines

$$[x,y] = P\exp\left(ig\int_x^y dz^\mu A_\mu(z)\right)$$

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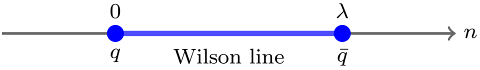
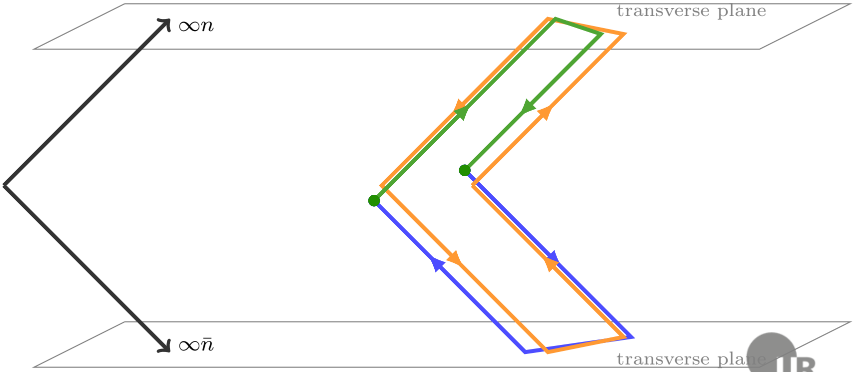


$n^2 = 0$
on light-cone

TMD factorization is full of (light-like) Wilson lines

$$\frac{d\sigma}{dX} \simeq H(Q) \int \frac{d^2 b}{(2\pi)^2} e^{i(bk)_T} \, f(x_A, b) \, S(b) \, f(x_B, b)$$



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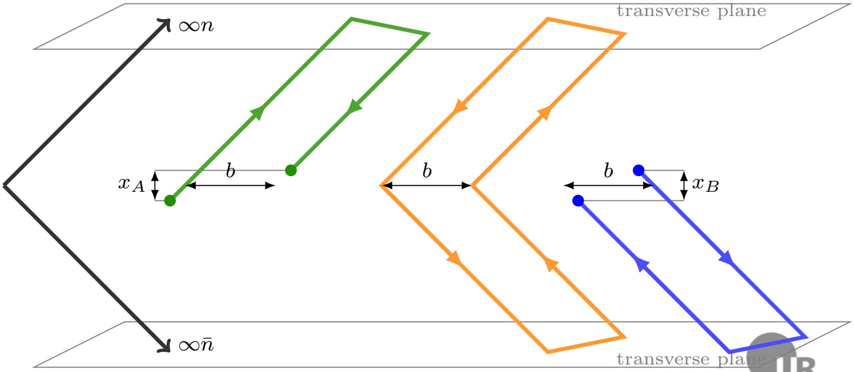
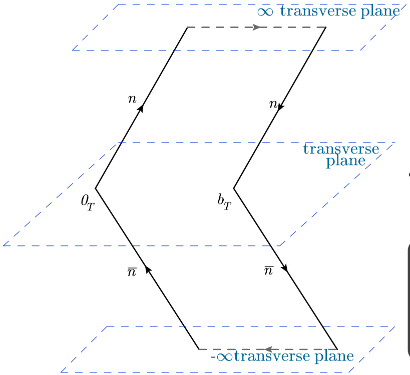


TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n \cdot \bar{n}) = 1$$

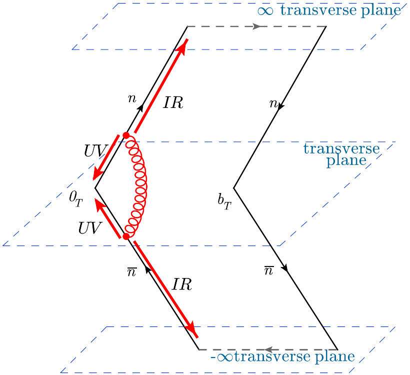
Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Looks simple, but SF is a theoretician's nightmare.
Multiple divergences!

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}^\dagger_n(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$

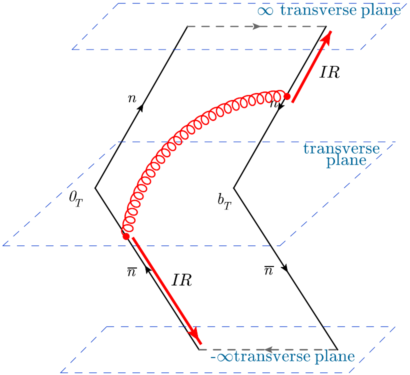


$$\begin{aligned}\int \quad dxdy \quad & D(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-} \\ = \quad & \int_0^\infty \frac{dx^+}{x^+} \int_0^\infty \frac{dy^-}{y^-} \\ = \quad & (\mathrm{UV}+\mathrm{IR})\,(\mathrm{UV}+\mathrm{IR})\end{aligned}$$

Some people set it to zero.

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned}\int \quad dxdy \quad D(x-y) \\ &= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)} \\ &= \text{IR at } x,y\to\infty\end{aligned}$$

However, it exactly cancels IR from the previous diagram
IR-cancellation proved at all orders,
e.g.[Echevarria,Scimemi,AV,1511.05590]

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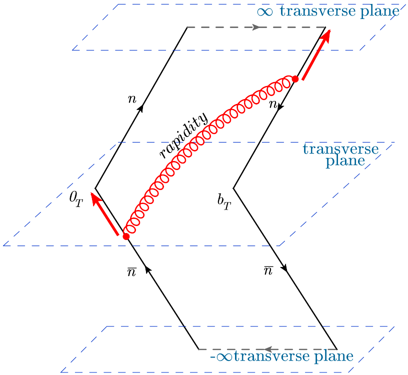
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \underbrace{\int_0^\infty \frac{d\sigma}{\sigma}}_{\text{rap.div}} \underbrace{\int_0^\infty \frac{dLL}{(2L^2+\mathbf{b}^2)}}_{\text{IR}}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

Regularizations for rapidity divergences

- Rapidity divergences are not regularized by dim.reg.
- There are many regularizations:
  - δ-regularization [Echevarria,Scimemi,AV,1511.05590],
  - exponential-regularization [Li,Neill,Zhu,1604.00392],
  - off-light-cone Wilson lines [Collins' textbook],
  - analytical regularization [Chiu, et al,1104.0881],
  - ...

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$ (2-loop check [1511.05590])

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

$\delta^{+(-)}$ regularizes rap.div. in $n(\bar{n})$ direction, $\delta^{\pm} \to 0$.

- Important note 1: the structure holds for arbitrary ϵ
- Important note 2: the structure holds at all orders of PT [AV,1707.07606]

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Factorization of rapidity-divergences

$$S(b_T) = \exp\big(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\big)$$

$$= \exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\zeta^+(\delta^+)^2) + \frac{B(b_T,\epsilon)}{2}\right)\exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\zeta^-(\delta^-)^2) + \frac{B(b_T,\epsilon)}{2}\right)$$

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Factorization of rapidity-divergences

$$S(b_T) = \exp\big(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\big)$$

$$= \exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\underset{\substack{\uparrow\\ \text{rap.div.}\\ \text{in } n\text{-direction}}}{\zeta^+}(\delta^+)^2) + \frac{B(b_T,\epsilon)}{2}\right)\exp\left(\frac{A(b_T,\epsilon)}{2}\ln(\underset{\substack{\uparrow\\ \text{rap.div.}\\ \text{in } \bar n\text{-direction}}}{\zeta^-}(\delta^-)^2) + \frac{B(b_T,\epsilon)}{2}\right)$$

- The rapidity divergences related to different sectors factorize
- Factorization introduces an additional scales ζ: (here $\zeta^+\zeta^- = 1$)
- The factorization of rapidity divergences in TMD soft-factor is a consequence of **renormalization theorem for rapidity divergences.**

Renormalization theorem for rapidity divergences.[AV,1707.07606]

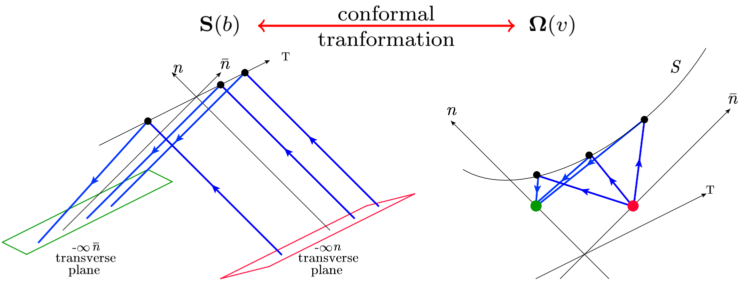
At *any finite order* of perturbation theory there exists the "rapidity divergence renormalization factor" $\mathbf{R}_n$, which contains only rapidity divergences associated with the direction n, such that the combination

$$S^R(\{b\},\zeta^+,\zeta^-)=\mathbf{R}_n(\{b\},\zeta^+)S(\{b\})\mathbf{R}_{\bar{n}}^\dagger(\{b\},\zeta^-)$$

is free of rapidity divergences.

The essence of proof is the equivalence of rapidity divergences and ultraviolet divergences.

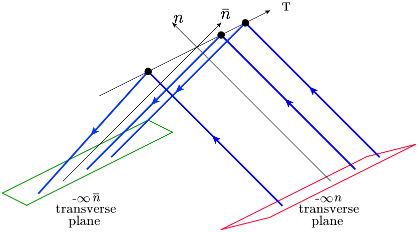
In conformal field theory

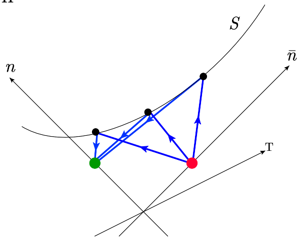


The essence of proof is the equivalence of rapidity divergences and ultraviolet divergences.

In conformal field theory

$$\mathbf{R}(b)\mathbf{S}(b)\mathbf{R}^\dagger(b) \xleftrightarrow[\text{tranformation}]{\text{conformal}} \mathbf{Z}(v)\mathbf{\Omega}(v)\mathbf{Z}^\dagger(v)$$





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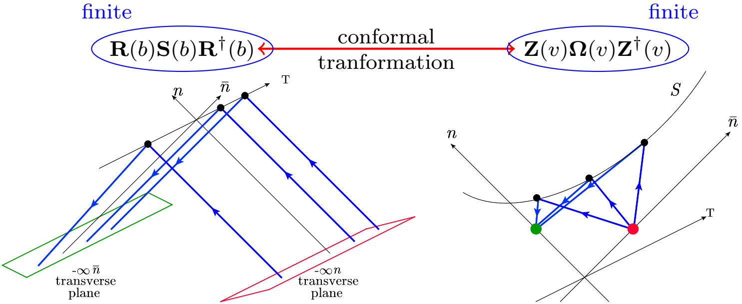
In conformal field theory



In CFT rapidity renormalization factor equals to UV renormalization

In QCD

The existence of renormalization can be proved order-by-order with iterations using

- Renormalization statement in CFT
- Conformal-invariance of QCD at tree-order.

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In QCD

The existence of renormalization can be proved order-by-order with iterations using

- Renormalization statement in CFT
- Conformal-invariance of QCD at tree-order.

What I did not discuss

- Factorization for Multi-Parton scattering
- All-order restrictions on the soft anomalous dimension
- Counting rules for rap.div., and definition of R'-operation
- Overlap of rap.divs. and violation of factorization

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Rapidity anomalous dimension

Similarly to UV renormalization the rapidity-divergence renormalization satisfies **renormalization group equation** with respect to rapidity divergence renormalization scale ζ. The scaling with ζ has anomalous dimension

$$\mathcal{D}(b) = \frac{1}{2}\mathbf{R}^{-1}(b,\zeta)\frac{d}{d\ln\zeta}\mathbf{R}(b,\zeta)$$

In literature, this object is known

- under different names: "non-perturbative Sudakov kernel", "CSS kernel", "rapidity anomalous dimension".
- under different letters $-K(b)/2$ [Collins,et al], $F_{q\bar{q}}(b)$ [Becher,Neubert], γ_ν, ...

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Important consequence:
correspondence between soft- and rapidity-anomalous dimensions.

In conformal field theory

$$\mathcal{D}(\mu, \mathbf{b}) = \gamma_s(\mu, (v_1 \cdot v_2)), \qquad (v_1 \cdot v_2) = \mathbf{b}^2 e^{2\gamma_E}/4$$

Checked by explicit calculation in $\mathcal{N} = 4$SYM [Li,Zhu,1604.01404]

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Important consequence:
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In QCD

In QCD the same relation holds at the critical point (e.g.$\epsilon^*=-\beta(a_s)$)

$$\mathcal{D}(\mu,\mathbf{b};\epsilon^*)=\gamma_s(\mu,(v_1\cdot v_2)),\qquad (v_1\cdot v_2)=\mathbf{b}^2e^{2\gamma_E}/4$$

This relation allows one to gain β-function terms of higher-order from lower order [AV,1610.05791].

Using this relation, one can derive 3-loop $\mathcal{D}$ from 2-loop $\mathcal{D}$ and 3-loop γ_s.

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$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

How to use it?

- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$
- $\epsilon^* = 0 - a_s\beta_0 - a_s^2\beta_1 - a_s^3\beta_2 - \ldots$
- We can compare order by order in PT

$$\begin{aligned}
\mathbf{D}_1(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_1(\{v\}),\\
\mathbf{D}_2(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_2(\{v\}) + \beta_0\mathbf{D}_1'(\{b\}),\\
\mathbf{D}_3(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_3(\{v\}) + \beta_0\mathbf{D}_2'(\{b\}) + \beta_1\mathbf{D}_1'(\{b\}) - \frac{\beta_0^2}{2}\mathbf{D}_1''(\{b\}),
\end{aligned}$$

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3-loop expression for RAD

$$\mathcal{D}_1(\mathbf{b}^2,\epsilon)=-2a_sC_F\Big[\Big(\frac{\mathbf{b}^2}{4}\Big)^{\epsilon}\Gamma(-\epsilon)+\frac{1}{\epsilon}\Big]=a_sC_F\Big\{2\mathbf{L}_\mu+\epsilon\underbrace{(\mathbf{L}_\mu^2+\zeta_2)}_{D_1'}+...\Big\}$$

$$\begin{aligned}\mathcal{D}_2(\mathbf{b}^2,\epsilon) &= a_s^2C_F\Big\{\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)\\ &+\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\Big(\frac{3(4-3\epsilon)}{2\epsilon}C_A-N_f\Big)\Big)+\boldsymbol{B}^\epsilon\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0+\frac{\beta_0}{2\epsilon^2}-\frac{\Gamma_1}{2\epsilon}\Big\}\end{aligned}$$

Taking

$$\gamma_s=C_Fa_s\left(\Gamma_0\mathcal{L}_\mu-\tilde\gamma_0\right)+C_Fa_s^2\left(\Gamma_1\mathcal{L}_\mu-\tilde\gamma_1\right)+C_Fa_s^3\left(\Gamma_2\mathcal{L}_\mu-\tilde\gamma_2\right)+...$$

We find

$$\mathcal{D}_3(\mathbf{b}^2,0)=\text{logs}-\frac{\tilde\gamma_2}{2}+(\beta_1+\beta_0\Gamma_1)\zeta_2-\frac{2}{3}\beta_0^2\zeta_3+\beta_0\Big\{C_A\left(\frac{2428}{81}-26\zeta_4\right)-N_f\frac{328}{81}\Big\}$$

It coincides with the direct calculation [Li,Zhu,1604.01404].

$$
\begin{aligned}
\mathcal{D}^{(3)}_{L=0} \quad = \quad & -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3 - \frac{88}{3}\zeta_2\zeta_3 - 192\zeta_5 - \frac{297029}{729} + \frac{6392}{81}\zeta_2 + \frac{154}{3}\zeta_4\right) \\
& -\frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3 + \frac{62626}{729} - \frac{824}{81}\zeta_2 + \frac{20}{3}\zeta_4\right) - \\
& \frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3 + \frac{1711}{27} - 16\zeta_4\right) - \frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3 - \frac{1856}{729}\right)
\end{aligned}
$$

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Important

in QCD

rapidity anomalous dimension is generically non-perturbative

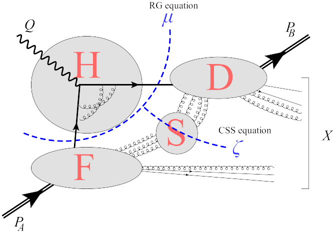
- Non-perturbative terms important at $b \gtrsim \Lambda_{\text{QCD}}^{-1}$
- At $b \to 0$ is entirely perturbative
- There is no "non-perturbative" proof of factorization, but it is expected (e.g. b^2-correction factorizes at LO [Scimemi,AV,1609.06047], at all orders [AV,in prep.])
- All non-perturbative correction must turn to zero at ϵ^*, "renomalon nature".

Final form of TMD factorization

$$d\sigma \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2) \;\; \Phi_{h_1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

spliting rapidity singularities
$S(b_T) \to R(b_T, \zeta^+) \cdot S_0 \cdot R^\dagger(b_T, \zeta^-)$

$$d\sigma \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\; F(z_1, b_T; \zeta^+)\; D(z_2, b_T; \zeta^-) + Y$$



TMD PDF
$R\Phi_{h_1}$
(regular)

TMD FF
$R\Delta_{h_2}$
(regular)

S_0 is the finite parts
of rapidity renormalization
Commonly used
renormalization scheme: $S_0 = 1$

Part II:
TMD evolution

$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T} = \sigma_0 \int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1,b;\mu,\zeta_1) D_{f'\leftarrow h}(x_2,b;\mu,\zeta_2) + ...$$

Evolution

TMD evolution is given by 2 equations

$$\mu^2 \frac{dF(x,b;\mu,\zeta)}{d\mu^2} = \gamma_F(\mu,\zeta)F(x,b;\mu,\zeta), \qquad \zeta \frac{dF(x,b;\mu,\zeta)}{d\zeta} = -\mathcal{D}(\mu,b)F(x,b;\mu,\zeta)$$

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$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T}=\sigma_0\int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)}H_{ff'}(Q,\mu)F_{f\leftarrow h}(x_1,b;\mu,\zeta_1)D_{f'\leftarrow h}(x_2,b;\mu,\zeta_2)+...$$

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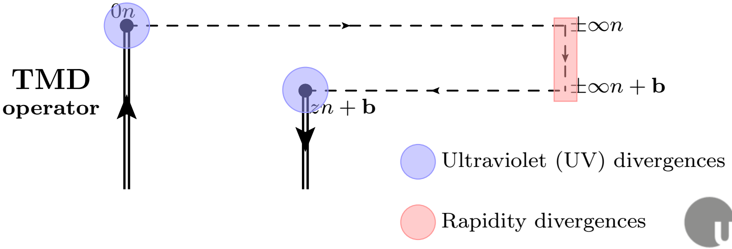
$$\mu^2\frac{dF(x,b;\mu,\zeta)}{d\mu^2}=\gamma_F(\mu,\zeta)F(x,b;\mu,\zeta),\qquad \zeta\frac{dF(x,b;\mu,\zeta)}{d\zeta}=-\mathcal{D}(\mu,b)F(x,b;\mu,\zeta)$$



TMD evolution is a double-scale evolution

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \tag{1}$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \tag{2}$$

Both anomalous dimensions related to each other (CS equation [Collins,Sopper,1981])

$$-\zeta \frac{d\gamma_F(\mu,\zeta)}{d\zeta} = 2\mu^2 \frac{d\mathcal{D}(\mu,b)}{d\mu^2} = \Gamma_{\mathrm{cusp}}(\mu) \tag{3}$$

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TMD evolution is a double-scale evolution

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

Solution: $F(x,\mathbf{b};\mu_f,\zeta_f) = R[\mathbf{b};(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,\mathbf{b};\mu_i,\zeta_i)$

Expression for R is known as "Sudakov exponent"

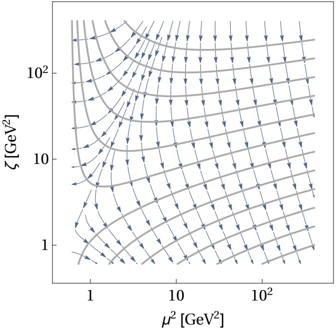
e.g.[Collins' textbook]

$$\times \exp\left\{ \ln \frac{\sqrt{\zeta_A}}{\mu_b} \tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_D(g(\mu');1) - \ln \frac{\sqrt{\zeta_A}}{\mu'} \gamma_K(g(\mu')) \right] \right\}. \quad (13.70)$$

This is probably the best formula for calculating and fitting TMD fragmentation functions;

Two-dimensional picture

see details in [Scimemi,AV,1803.11089]



TMD evolution is 2D evolution

$$\mu^2 \frac{dF(x,b;\mu,\zeta)}{d\mu^2} = \gamma_F(\mu,\zeta)F(x,b;\mu,\zeta)$$

$$\zeta \frac{dF(x,b;\mu,\zeta)}{d\zeta} = -\mathcal{D}(\mu,b)F(x,b;\mu,\zeta)$$

or

$$\vec{\boldsymbol{\nabla}} F = \vec{\mathbf{E}} F$$

Evolution field	$\mathbf{E} = \left(\frac{\gamma_F}{2}, -\mathcal{D}\right)$
is conservative	$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$
Evol.potential:	$\mathbf{E} = \boldsymbol{\nabla} U$

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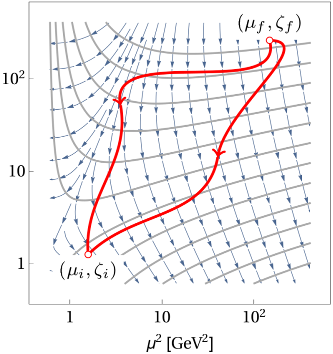
Two-dimensional picture

see details in [Scimemi,AV,1803.11089]



TMD evolution is 2D evolution

$$R[\mathbf{b}; i \to f] =$$
$$\exp \int_P d\boldsymbol{\nu} \cdot \mathbf{E} = \exp(U_f - U_i) =$$
$$\exp \Big[\int_P \Big(\gamma_F(\mu,\zeta) \frac{d\mu}{\mu} - \mathcal{D}(\mu,\mathbf{b}) \frac{d\zeta}{\zeta} \Big) \Big]$$

- Path independence
- Unified picture of various evolution scenarios

Evolution field	$\mathbf{E} = \left(\frac{\gamma_F}{2}, -\mathcal{D}\right)$
is conservative	$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$
Evol.potential:	$\mathbf{E} = \boldsymbol{\nabla} U$

Solution

$$F(x, \mathbf{b}; \mu_1, \zeta_1) = R[\mathbf{b}; (\mu_1, \zeta_1) \to (\mu_2, \zeta_2)]F(x, \mathbf{b}; \mu_2, \zeta_2)$$

$$\boxed{\begin{array}{c}\textbf{Initial scales:}\\ \mu_1 \simeq Q\\ \zeta_1 = Q^2\end{array}} \longrightarrow \boxed{\begin{array}{c}\textbf{Final scales:}\\ \mu_2 \sim \ ??\\ \zeta_2 \sim \ ??\end{array}}$$

Solution

$$F(x,\mathbf{b};\mu_1,\zeta_1) = R[\mathbf{b};(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)]F(x,\mathbf{b};\mu_2,\zeta_2)$$

Initial scales:
$\mu_1 \simeq Q$
$\zeta_1 = Q^2$

$\longrightarrow$

Final scales:
$\mu_2 \sim\ ??$
$\zeta_2 \sim\ ??$

Main complication:
$\mathbf{b}^2 \in (0,\infty)$
perturbative logarithms $\ln(\mathbf{b}^2\mu^2)$, $\ln(\mathbf{b}^2\zeta)$, $\ln(\mu^2/\zeta)$, $a_s(\mu)$

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Solution

$$F(x,\mathbf{b};\mu_1,\zeta_1) = R[\mathbf{b};(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)]F(x,\mathbf{b};\mu_2,\zeta_2)$$

Initial scales:
$\mu_1 \simeq Q$
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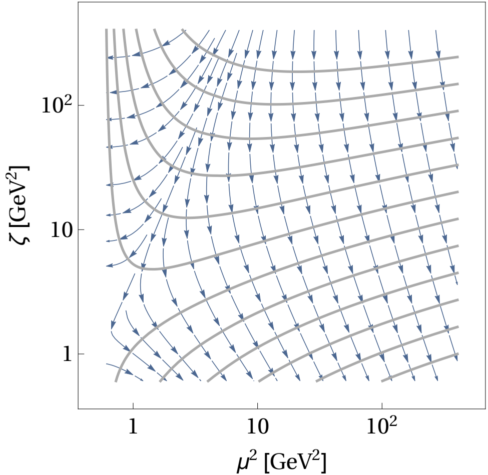
$\longrightarrow$

Final scales:
$\mu_2 \sim$??
$\zeta_2 \sim$??

Main complication:
$\mathbf{b}^2 \in (0,\infty)$
perturbative logarithms $\ln(\mathbf{b}^2\mu^2)$, $\ln(\mathbf{b}^2\zeta)$, $\ln(\mu^2/\zeta)$, $a_s(\mu)$

It is not possible to minimize all logarithms in a reasonably wide range of b.
Something blows up in any case.
But it is not needed!

TMD distribution is not defined by a scale (μ, ζ)
It is defined by an equipotential line.



The scaling is defined by
~~a difference between scales~~
a difference between potentials

TMD distribution is not defined by a scale (μ, ζ)
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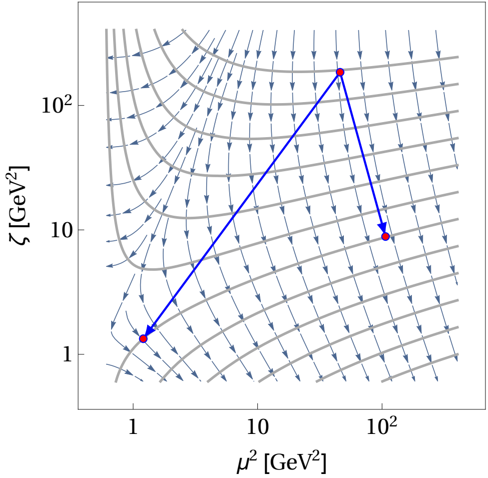

The scaling is defined by
~~a difference between scales~~
a difference between potentials

Evolution factor to both points
is the same
although the scales are
different by 10^2GeV^2

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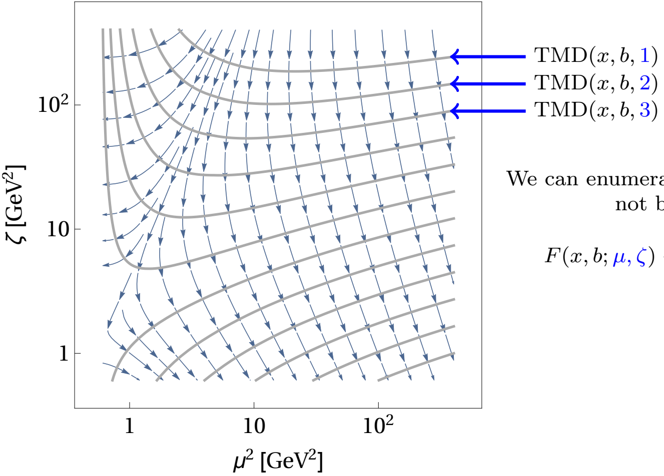
TMD distributions on the same equipotential line are equivalent.



We can enumerate them by a lines not by (μ, ζ)

$$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$$

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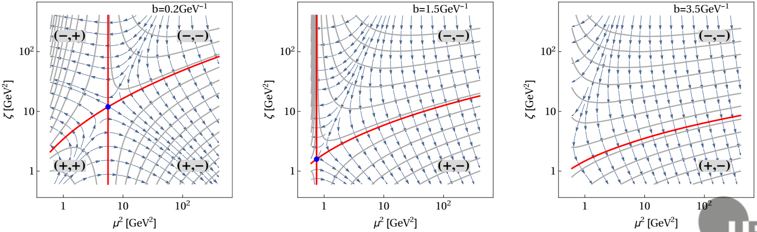
There is a unique line which passes though all μ's

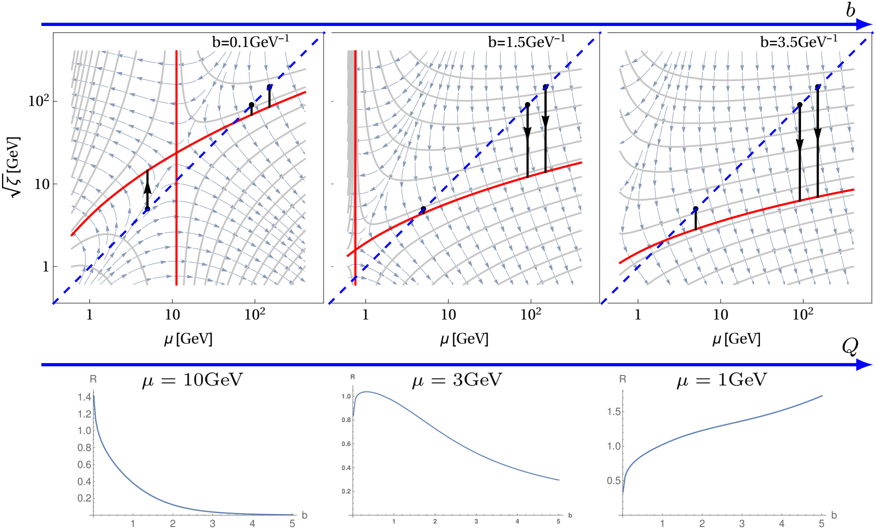
The optimal TMD distribution

$$F(x,b) = F(x,b;\mu,\zeta_\mu)$$

where ζ_μ is the special line.



The evolution potential depends on b.
Relative position of its elements (saddle-point, special lines) dictates the shape of evolution factor.



$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T} = \sigma_0 \int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1,b;\mu,Q^2) F_{f'\leftarrow h}(x_2,b;\mu,Q^2) + \ldots$$

Evolution

$$\frac{d\sigma}{dydQ^2d^2\mathbf{q}_T} = \sigma_0 \int d^2b\, e^{i(\mathbf{b}\cdot\mathbf{q}_T)} H_{ff'}(Q,\mu) R[\mathbf{b};(\mu,Q^2)\to \text{s.l.}]^2 F_{f\leftarrow h}(x_1,b) F_{f'\leftarrow h}(x_2,b) + \ldots$$

Evolution factor has simple expression

$$R[\mathbf{b};(\mu,\zeta)\to \text{s.l.}] = \left(\frac{\zeta}{\zeta_\mu}\right)^{-\mathcal{D}(\mathbf{b},\mu)}$$

Good PT convergence

$\mu = Q$

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Part III:
Practice

Single cross-section ⇒ three NP functions

$$\frac{d\sigma}{dp_T^2 dQ} \simeq \sigma_0(Q) \int d^2\mathbf{b} e^{i\mathbf{b}\mathbf{p}_T} \left(\frac{Q^2}{\zeta_Q(b)}\right)^{-2\mathcal{D}(Q,b)} F_1(x_1,\mathbf{b}) F_2(x_2,\mathbf{b})$$

- TMD distribution $F_1(x_1, b)$
- TMD distribution $F_2(x_2, b)$
- non-perturbative evolution $\mathcal{D}(Q, b)$

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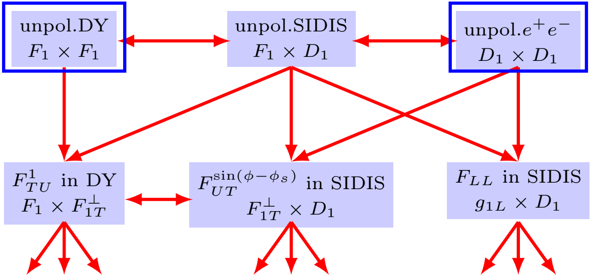
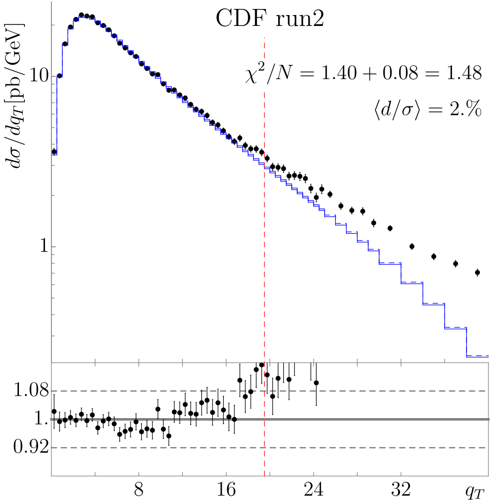
Single cross-section $\Rightarrow$ three NP functions

$$\frac{d\sigma}{dp_T^2 dQ} \simeq \sigma_0(Q) \int d^2\mathbf{b} e^{i\mathbf{b}\mathbf{p}_T} \left(\frac{Q^2}{\zeta_Q(b)}\right)^{-2\mathcal{D}(Q,b)} F_1(x_1,\mathbf{b}) F_2(x_2,\mathbf{b})$$

- TMD distribution $F_1(x_1, b)$
- TMD distribution $F_2(x_2, b)$
- non-perturbative evolution $\mathcal{D}(Q, b)$



TMD factorization $\Rightarrow$ small-q_T/Q



Data-cut rule

$$q_T \simeq 0.25Q$$

see study in [I.Scimemi,AV,1706.01473]

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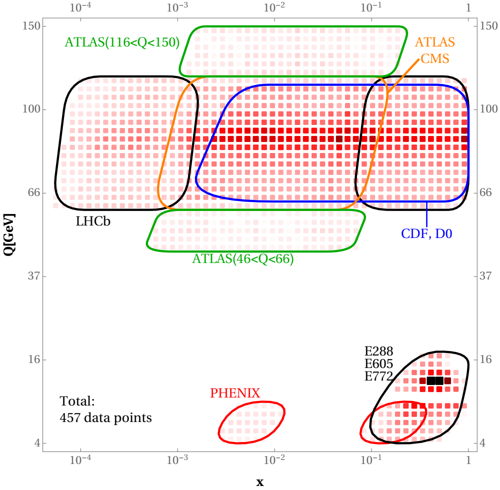
Global fit of TMD Drell-Yan data

[V.Bertone,I.Scimemi,AV,1902.08474]



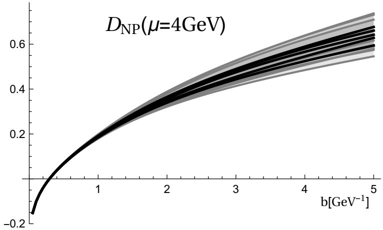
$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y} \sqrt{1 + \frac{q_T^2}{Q^2}}$$

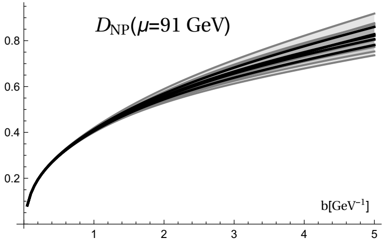
High-energy: CDF, D0, ATLAS, CMS, LHCb
194 points

Low-energy: E288, E605, E772, PHENIX
263 points

Total: 457 points
$4 < Q < 150\text{GeV}$
$x > 10^{-4}$

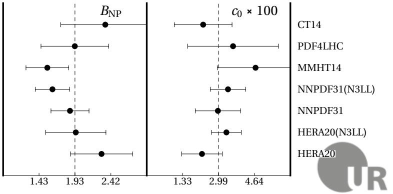
Non-perturbative evolution kernel





$$\mathcal{D}(\mathbf{b}) = \mathcal{D}_{\text{pert}}(b^*(\mathbf{b})) + c_0 \mathbf{b} b^*(\mathbf{b}), \qquad b^*(\mathbf{b}) = \mathbf{b}/\sqrt{1+\mathbf{b}^2/B_{\text{NP}}^2}$$

$$\boxed{\begin{array}{c} B_{\text{NP}} \simeq 2\text{GeV} \\ c_0 \simeq 0.03\text{GeV}^2 \end{array}}$$



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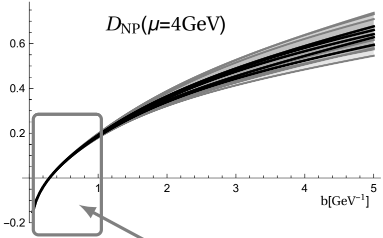
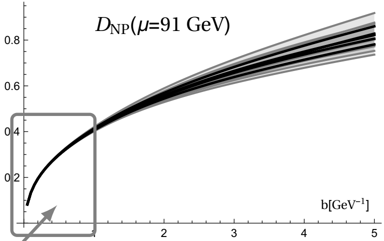
Non-perturbative evolution kernel

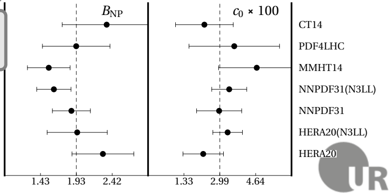




$$\mathcal{D}(\mathbf{b}) = \mathcal{D}_{\text{pert}}(b^*(\mathbf{b})) + c_0 \mathbf{b} b^*(\mathbf{b}), \qquad b^*(\mathbf{b}) = \mathbf{b}/\sqrt{1 + \mathbf{b}^2/B_{\text{NP}}^2}$$

LHC
sensitive region

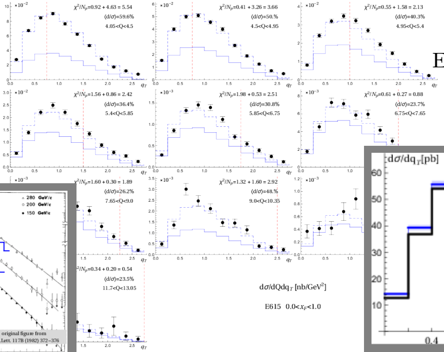
$B_{\text{NP}} \simeq 2$
$c_0 \simeq 0.03$GeV

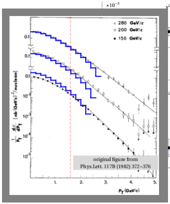


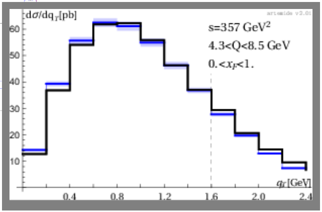
Extracted non-perturbative rapidity anomalous dimension
is universal and can be used to describe different data

Pion-induced Drell-Yan [AV,1907.10356]

E537, E615, NA3
$\chi^2/(N_p = 80) = 1.44$
E615 normalization issue?







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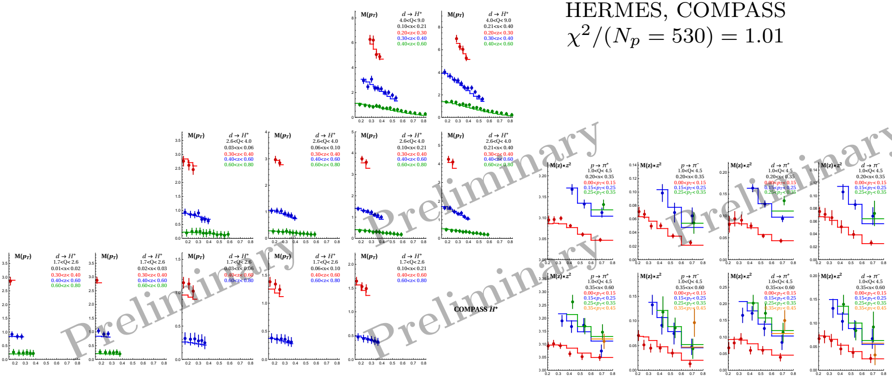
Extracted non-perturbative rapidity anomalous dimension is universal and can be used to describe different data

Semi-inclusive deep inelastic scattering [Scimemi,AV,work in progress]

HERMES, COMPASS

$\chi^2/(N_p = 530) = 1.01$



Extracted non-perturbative rapidity anomalous dimension
is universal and can be used to describe different data

`artemide`

Program package for TMD phenomenology

- Efficient code based on TMD factorization
- Variety of evolution schemes (CSS, ζ-prescription, improved-$\mathcal{D}$, resummed, etc)
- Full control on non-perturbative and model inputs.
- All possible combinations of perturbative inputs LO,NLO,NNLO,"NNNLL"
- Bin-integrations, lepton cuts, etc.
- The library of processes is constantly updating
  - Drell-Yan -like
    - (unpolarized) Z/γ^*, W's, Higgs, pion-induces.
  - SIDIS
    - unpolarized
    - Sivers effect (in preparation)
  - More in plans

repository:
https://github.com/VladimirovAlexey/artemide-public

Part IV:
A bit on interpretation

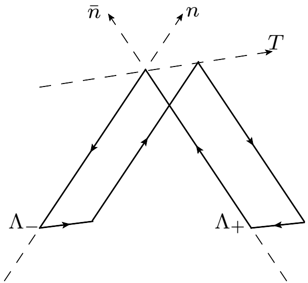
(in progress)

Rapidity anomalous dimension is function that directly measures properties of QCD vacuum
Which properties?

Non-perturbative definition of RAD

Rapidity anomalous dimension is independent on regularization.

$$S(b;\Lambda_+\Lambda_-)=\frac{\mathrm{Tr}}{N_c}\langle 0|P\exp\left(-ig\int_C dx^\mu A_\mu(x)\right)|0\rangle$$



At $\Lambda_+\Lambda_-\to\infty$

$$S(b;\Lambda_+\Lambda_-)=\exp\left(-2\mathcal{D}(b)\ln(\Lambda_+\Lambda_-)+....\right)$$

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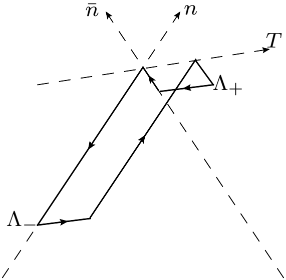
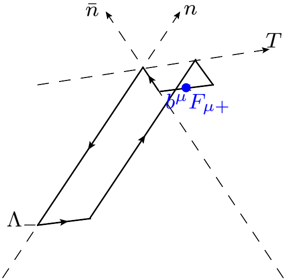


At $\Lambda_+\Lambda_-\to\infty$

$$S(b;\Lambda_+\Lambda_-)=\exp\left(-2\mathcal{D}(b)\ln(\Lambda_+\Lambda_-)+....\right)$$

Due to Lorentz invariance
it is enough to $\Lambda_-\to\infty$

Rapidity anomalous dimension can be defined as a primary object.



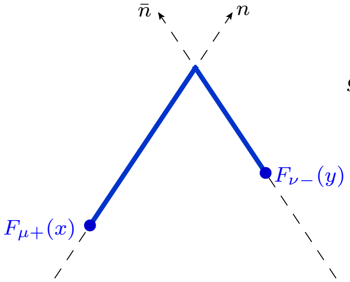
$$\mathcal{D}(b) = \frac{1}{2}\lim_{\Lambda_-\to\infty}\frac{S'(b;\Lambda_+\Lambda_-)}{S(b;\Lambda_+\Lambda_-)}$$

$$S'(b) = ig\int_0^1 d\beta \frac{\mathrm{Tr}}{N_c}\langle 0|F_{b+}(-\Lambda_+ b+\beta b)P\exp\left(-ig\int_{C'}dx^\mu A_\mu(x)\right)|0\rangle$$

- Route to non-perturbative calculation and modeling.

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + ...$$



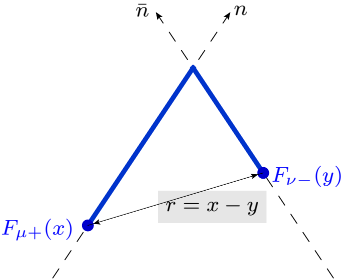
$\mathcal{D}_1$ is expressed via 2-point correlators connected by "a minimal distance link"

$$g^2\frac{\text{Tr}}{N_c}\langle 0|F_{\mu x}(x)[x,0][0,y]F_{\nu y}(y)|0\rangle =$$

$$\left(g^{\mu\nu} - \frac{y^\mu x^\nu}{(xy)}\right)\varphi_1(x,y) + (...)^{\mu\nu}\varphi_2$$

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + ...$$



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$$g^2\frac{\text{Tr}}{N_c}\langle 0|F_{\mu x}(x)[x,0][0,y]F_{\nu y}(y)|0\rangle =$$
$$\left(g^{\mu\nu} - \frac{y^\mu x^\nu}{(xy)}\right)\varphi_1(x,y) + (...)^{\mu\nu}\varphi_2$$

At LO $x^2 = y^2 = 0$
$\varphi_1(x,y) = \varphi_1(r^2)$
$\varphi_2(x,y) = 0$

$$\mathcal{D}_1(\mathbf{b}) = \frac{1}{2}\int_0^\infty \frac{d\mathbf{r}^2}{\mathbf{r}^2}\varphi_1(\mathbf{r}^2), \qquad \text{here } \mathbf{r}^2 = -r^2 > 0.$$

Power correction to $\mathcal{D}$

$$\mathcal{D}(\mathbf{b}) = \underbrace{\mathcal{D}_{\text{pert}}(\ln(\mu^2\mathbf{b}^2))}_{\text{known at N}^3\text{LO}} + \mathbf{b}^2\mathcal{D}_1(\ln(\mu^2\mathbf{b}^2)) + \mathbf{b}^4\mathcal{D}_2(\ln(\mu^2\mathbf{b}^2)) + ...$$

Estimation

$$\mathcal{D}_1(\mathbf{b}) = \frac{1}{2}\int_0^\infty \frac{d\mathbf{r}^2}{\mathbf{r}^2}\varphi_1(\mathbf{r}^2), \qquad \text{here } \mathbf{r}^2 = -r^2 > 0.$$

- At $\mathbf{r}^2 \to 0$, $\varphi_1 \sim \mathbf{r}^2 \frac{\pi^2}{36} G_2$
- At $\mathbf{r}^2 \to \infty$, $\varphi_1 \sim \frac{1}{\mathbf{r}^2}$ (at least)

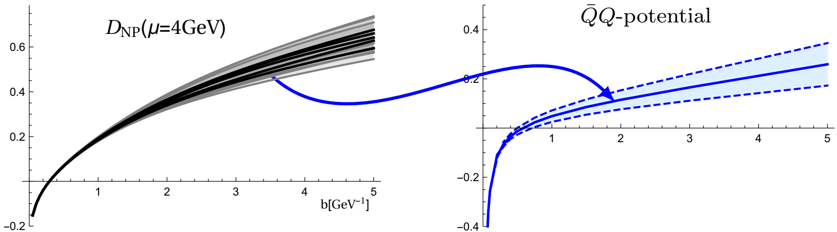
So, order-of-magnitude estimation

$$\mathcal{D}_1 \lesssim \frac{\pi^2}{72}\frac{G_2}{\Lambda^2_{\text{QCD}}} \sim (0.01 - 0.05)\text{GeV}^2$$

Extracted value

$$\mathcal{D}_1 = 0.022 \pm 0.009\text{GeV}^2$$

Non-perturbative evolution kernel
measures properties of QCD vacuum
(but requires model for interpretation)



Stochastic Vacuum model

- The simplest model for QCD vacuum (Wilson-lines **unimportant**)
- Allows for the definition of a "static potential" (linear)

$$V(\mathbf{r}) \quad = \quad \mathbf{r}\frac{\pi}{4}\mathcal{D}''(0) + \frac{\mathcal{D}'(0)}{2} + \frac{\mathbf{r}^2}{2}\int_{\mathbf{r}}^{\infty} d\mathbf{x}\frac{\mathcal{D}'(\mathbf{x})}{\mathbf{x}^2\sqrt{\mathbf{x}^2-\mathbf{r}^2}}.$$

- "String tension" $\sigma = \frac{\pi}{4}\mathcal{D}''(0) = \frac{\pi}{2}c_0 \simeq 0.05 \pm 0.02\text{GeV}^2$ vs. 0.19GeV^2

Conclusion

Nowadays,

- transverse momentum dependent (TMD) factorization theorem **is proved**,
- divergences of (TMD) soft factors **are understood**,
- rapidity anomalous dimension **are known at NNLO**
- TMD evolution **works!**.

Future

- global phenomenology DY+SIDIS, asymmetries
- interpretation and models,
- lattice measurements of $\mathcal{D}$,
- ...