Renormalization theorem for rapidity divergences & rapidity anomalous dimension

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based on [1707.07606]

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The talk is about

- transverse momentum dependent (TMD) factorization theorems,
- soft factors,
- rapidity divergences,
- rapidity anomalous dimension and TMD evolution,
- correspondence between different processes,
- and higher-loop calculations without loop-diagrams.

Plan of the talk

- introduction to TMD factorization,
- geometric/spatial definition of divergences,
- proof of renormalization theorem for rapidity divergences,
- correspondence between UV and rapidity divergences.

The modern factorization theorems have the following **general structure**

$$\underbrace{\frac{d\sigma}{dX}}_{cross-X} = \underbrace{H}_{\substack{\text{Hard part}\\ \text{perturbative}}} \times \underbrace{f_1 \otimes ... \otimes J_2}_{\substack{\text{Parton distributions}\\ \text{jet-functions, etc}\\ \text{Non-pertrubative}\\ \text{universal}}} \times \underbrace{S}_{\substack{\text{Soft factor(s)}\\ \text{perturbative ?}}} + \substack{\text{Some power}\\ \text{suppressed terms}}$$

- This is a typical result of field mode separation (SCET)
- Often, individual terms in the product are singular, and require "refactorization"

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In fact, some of parts of this construction are not proven/accurately formulated,

$\Rightarrow$ problems for higher calculation of perturbation theory

$\Rightarrow$ lack of "non-perturbative" definition for "non-perturbative functions"

$\Rightarrow$ absence of restrictions on the approach (working criterion)

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My talk is about factorization of soft factors and rapidity divergences, in TMD factorization theorems (but not only).

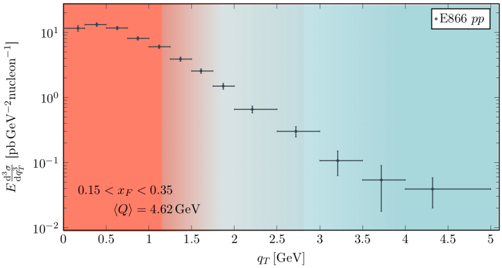
Introduction part I: TMD factorization

Transverse momentum dependent (TMD) factorization describes double-inclusive processes in the regime of small transverse momentum ($q_T^2 \ll Q^2$)

processes: $h_1 + h_2 \to \gamma^*/Z/W + X$ "Drell-Yan"

$h_1 + \gamma^* \to h_2 + X$ semi-inclusive DIS (SIDIS)

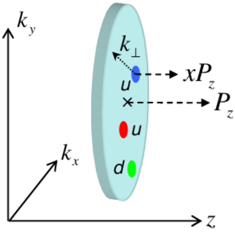
$e^+e^- \to h_1 + h_2 + X$

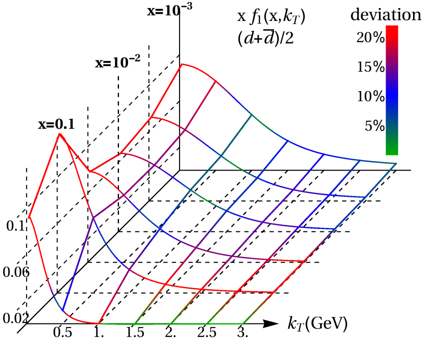


TMD regime collinear regime

The transverse momentum of photon q_T is determined with respect to "hadron plane"

In TMD regime the produced transverse momentum is mostly of "non-perturbative" origin: ⇒ TMD distributions (PDFs and FFs)





[Bertone,Scimemi,AV,in preparation]

TMD distributions should not be mistaken with collinear distributions (although they have some common points).

- Structurally different: different divergences and different evolution.

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

TMD factorization

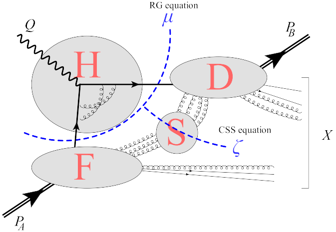
$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\ \ \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T) + Y$$

TMD soft factor

power suppressed terms

TMD FF

TMD PDF



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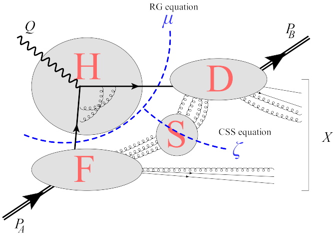
TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

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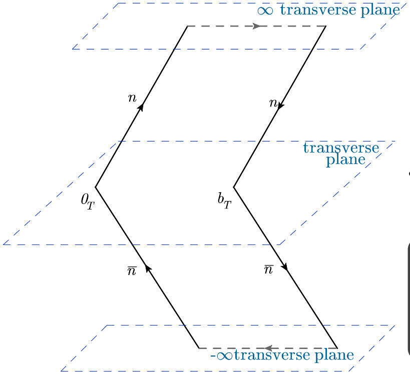
$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n \cdot \bar{n}) = 1$$

Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Looks simple, but SF is a theoretician's nightmare.
Multiple divergences!

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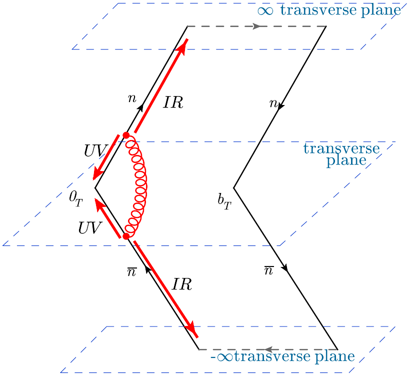
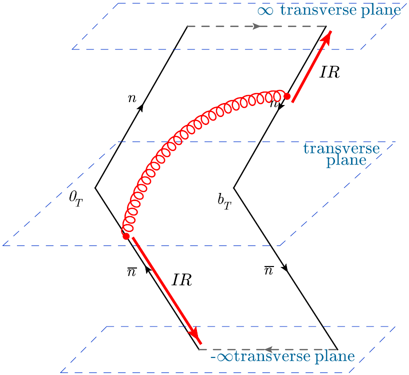
$$S(\mathbf{b}_T) = \langle 0|\text{Tr}\left(\boldsymbol{\Phi}_n(\mathbf{0}_T)\boldsymbol{\Phi}_n^\dagger(\mathbf{b}_T)\boldsymbol{\Phi}_{-n}(\mathbf{b}_T)\boldsymbol{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned}\int \quad dxdy \quad & D(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-} \\ = \quad & \int_0^\infty \frac{dx^+}{x^+} \int_0^\infty \frac{dy^-}{y^-} \\ = \quad & (\text{UV} + \text{IR})\,(\text{UV} + \text{IR})\end{aligned}$$

Some people set it to zero.

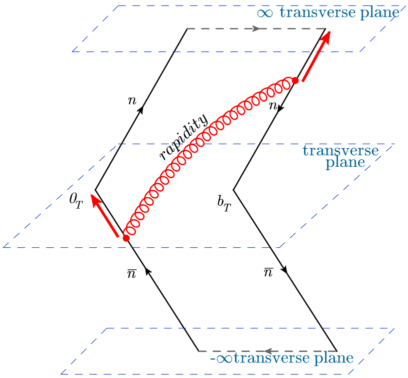
$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned}\int \quad dxdy \quad & D(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)} \\ = \quad & \text{IR at } x, y \to \infty\end{aligned}$$

However, it exactly cancels IR from the previous diagram
Proved at all orders,
e.g.[Echevarria,Scimemi,AV,1511.05590]

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\int dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \underbrace{\int_0^\infty \frac{d\sigma}{\sigma}}_{\text{rap.div}} \underbrace{\int_0^\infty \frac{dLL}{(2L^2+\mathbf{b}^2)}}_{\text{IR}}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

Regularizations for rapidity divergences

- Rapidity divergences are not regularized by dim.reg.
- There are many regularizations:
  - δ-regularization [Echevarria,Scimemi,AV,1511.05590],
  - exponential-regularization [Li,Neill,Zhu,1604.00392],
  - off-light-cone Wilson lines [Collins' textbook],
  - analytical regularization [Chiu, et al,1104.0881],
  - ...

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$ (2-loop check [1511.05590])

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

It allows to split rapidity divergences and define individual TMDs.

- Important note 1: the structure holds for arbitrary ϵ
- Important note 2: it is not obvious and will be proved here

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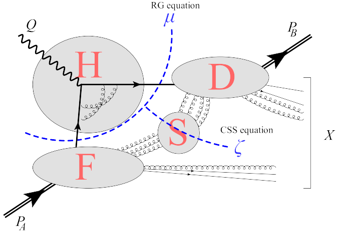
$$\exp(A\ln(\delta^+\delta^-)+B)=\exp\left(\frac{A}{2}\ln((\delta^+)^2\zeta)+\frac{B}{2}\right)\exp\left(\frac{A}{2}\ln((\delta^-)^2\zeta^{-1})+\frac{B}{2}\right)$$

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities
$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;F(z_1,b_T;\zeta^+)\;D(z_2,b_T;\zeta^-)+Y$$



TMD PDF
$\sqrt{S}\Phi_{h_1}$
(regular)

TMD FF
$\sqrt{S}\Delta_{h_2}$
(regular)

The extra "factorization" introduces an extra scale ζ.
And corresponded evolution equation

$$\zeta\frac{d}{d\zeta}F=\frac{A}{2}F=-\mathcal{D}F$$

Rapidity anomalous dimension (RAD)

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Introduction part II:
Double Drell-Yan scattering
&
multi-parton scattering soft factors

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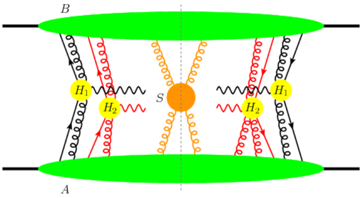


pictures from [1510.08696]

Double Drell-Yan scattering

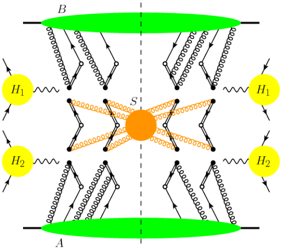
- Experimental status is doubtful
- Collinear part of factorization is proved [Diehl,et al,1510.08696]
- In many aspects similar to TMD factorization
- The same problem of rapidity factorization, but enchanted by the matrix structure

DPD soft factor (very singular)

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\ F^A_{h1}(z_{1,2}, b_{1,2,3,4}) S^{AB}(b_{1,2,3,4}) \bar{F}^B_{h2}(z_{1,2}, b_{1,2,3,4}) + Y$$

power suppressed terms

DPD (singular)



Structure is similar to TMD Drell-Yan but now it contains COLOR

The soft factor is a matrix

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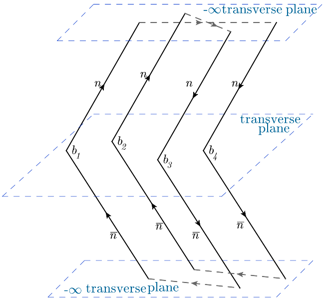
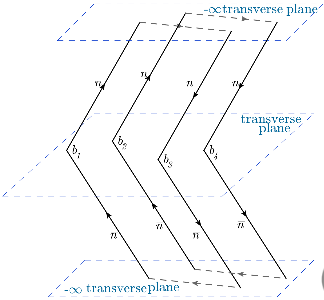
Color structure makes a lot of difference

$$F^A_{h1} S^{AB} \bar F^B_{h2} \xrightarrow{\text{singlets}} (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar F^{\mathbf{1}} \\ \bar F^{\mathbf{8}} \end{pmatrix}$$

- Soft-factors $S^{\mathbf{ij}}$ are sum of Wilson loops and double Wilson loops (all possible connections).
- Soft-factors are non-zero even in the integrated case.
- 2-loop calculation [AV,1608.04920]: rapidity divergences factorize (as a product of matrices) ⇒ matrix evolution equation



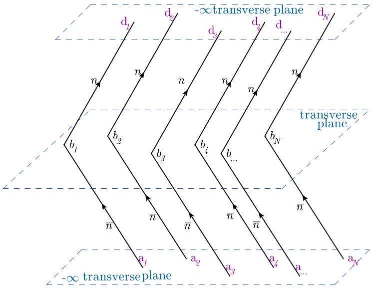


Let's look at multi-parton scattering

- Just as double-parton, but multi..(four WL's $\to$ arbitrary number WL's)
- Too many color-singlets, better to work with explicit color indices (color-multi-matrix)

$$\Sigma^{(a_1...a_N);(d_1...d_N)}(\mathbf{b}_1,...,\mathbf{b}_N)=\mathbf{\Sigma}(\mathbf{b}_{1,..,N})$$

$$\mathbf{\Sigma}(\{b\})=\langle 0|T\{[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_N)\dots[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_1)\}|0\rangle$$



Color-matrix notation

- All color flow in the same direction
- i'th WL has generator $\mathbf{T}_i$
- In total the soft factor is color-neutral

$$\sum_i \mathbf{T}_i=0.$$

- Color-neutrality $\to$ gauge invariance + cancellation IR singularities

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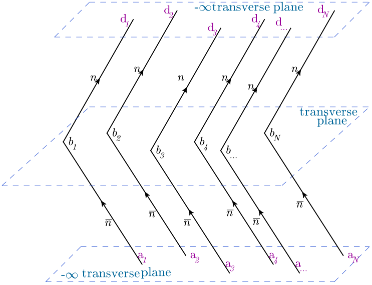

Result at NNLO is amazingly simple

$$\boldsymbol{\Sigma}(\mathbf{b}_{1,..,N})=\exp\left(-\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A\sigma(\mathbf{b}_{ij})+\mathcal{O}(a_s^3)\right)$$

- $\mathbf{T}_i^A\mathbf{T}_j^A$ = "dipole"
- $\mathcal{O}(a_s^3)$ contains also "color-multipole" terms
- Rapidity factorization for dipole part is straightforward (assuming TMD factorization)

These examples are parts of general picture, and could be described by single factorization/renormalization theorem.

Part I:
Renormalization theorem
for
rapidity divergences

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Rapidity divergences associated with different directions in the MPS soft factor could be factorized from each other. At *any finite order* of perturbation theory there exists the "rapidity divergence renormalization factor" $\mathbf{R}_n$, which contains only rapidity divergences associated with the direction n, such that the combination

$$\mathbf{\Sigma}^R(\{b\},\nu^+,\nu^-) = \mathbf{R}_n(\{b\},\nu^+)\mathbf{\Sigma}(\{b\})\mathbf{R}^\dagger_{\bar n}(\{b\},\nu^-)$$

is free of rapidity divergences.

- Implicitly, it has been expected for long time [Chiu,Jain,Neill,Rothstein,1104.0881]
- It is final block of the TMD factorization theorem (and also finalizes factorization for Double-DY)
- It has several non-trivial consequences.

Next, I am going to sketch the proof.

- Typically, such theorems are proved by considering singularities of Feynman diagrams.
- I will present a completely different approach.
- In fact, the approach could appear more interesting and important then the theorem it self.
- I will skip a lot of details, please, ask questions or look into [AV;1707.07606]

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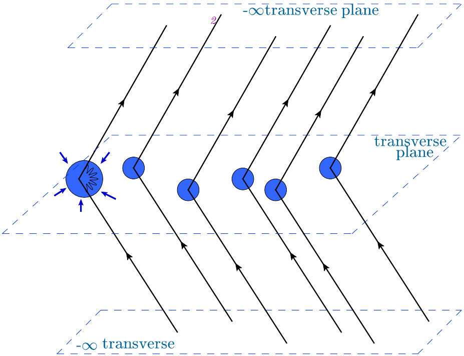
General picture of proof

- Isolate the spatial area of an operator which results into rapidity divergences.
- Invent a (conformal) transformation which map this area to a point (i.e. rapidity divergences to UV divergences)
- Using this transformation, and UV renormalization theorem, proof the theorem in CFT
- Generalize to QCD, using iteration procedure and restoration of conformal invariance at the QCD critical point.

Classification of divergences in coordinate space

Ultraviolet divergences (UV)



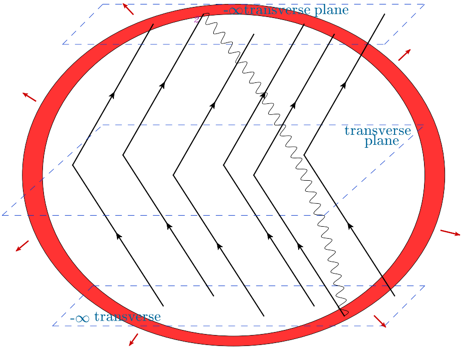
Localisation of fields in the vicinity of **a point**

$$x^2 \to 0$$

WARNING: depends on gauge fixation condition

Classification of divergences in coordinate space

Mass divergences (IR)



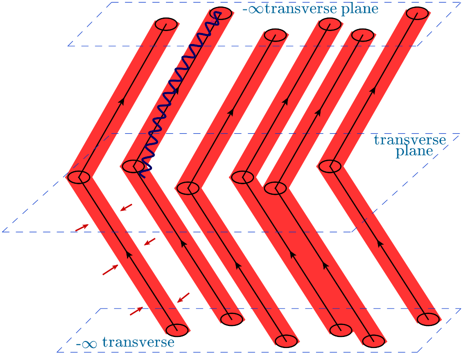
Localisation of fields at **the distant sphere**

$$x^2 \to \infty$$

WARNING: depends on gauge fixation condition

Classification of divergences in coordinate space

Collinear divergences (UV)



Localisation of fields in the vicinity of **a light-like line**

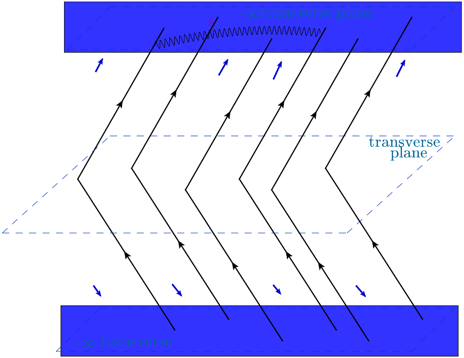
$(xp) \to 0 \quad (p^2 = 0)$

see better definition [Erdogan,Sterman,1411.4588]

WARNING: depends on gauge fixation condition

Classification of divergences in coordinate space

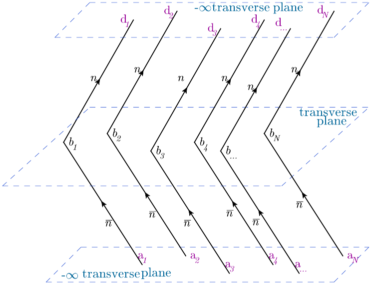
Ultraviolet divergences (UV)



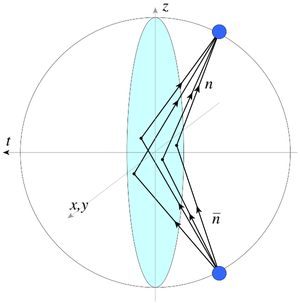
Localisation of fields in the vicinity of **a distant transverse plane**
see better definition [AV,1707.07606]

WARNING: depends on gauge fixation condition

Rapidity divergences associated with transverse planes (or better to say with the layer between the transverse plane and infinity). If we think of space-time as about a Riemann sphere, these planes are points at poles of Riemann sphere.



$\Longleftrightarrow$

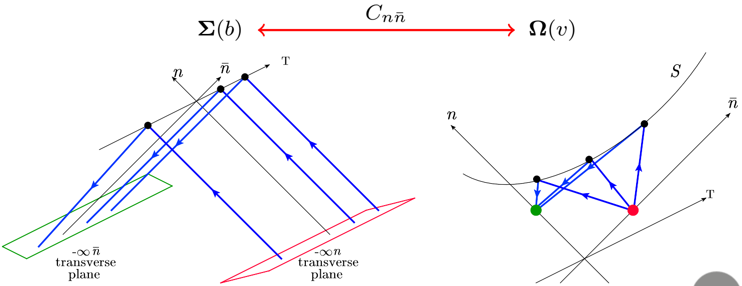


$$\mathcal{C}_{\bar{n}}:\{x^+,x^-,x_\perp\}\to\{\frac{-1}{2a}\frac{1}{\lambda+2ax^+},x^-+\frac{ax_\perp^2}{\lambda+2ax^+},\frac{x_\perp}{\lambda+2ax^+}\}$$

Composition of two conformal-stereographic transformations

$$C_{n\bar{n}}=\mathcal{C}_n\mathcal{C}_{\bar{n}}=\mathcal{C}_{\bar{n}}\mathcal{C}_n$$

With the special choice of parameters any DY-like soft factor transforms to a compact object.



In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization

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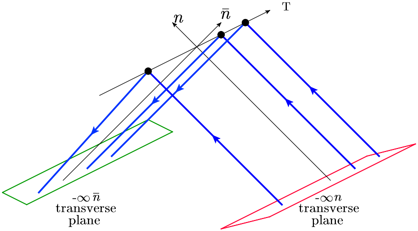
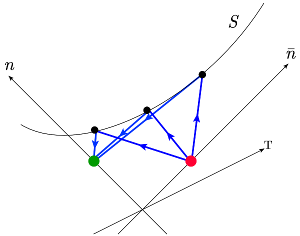
In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization

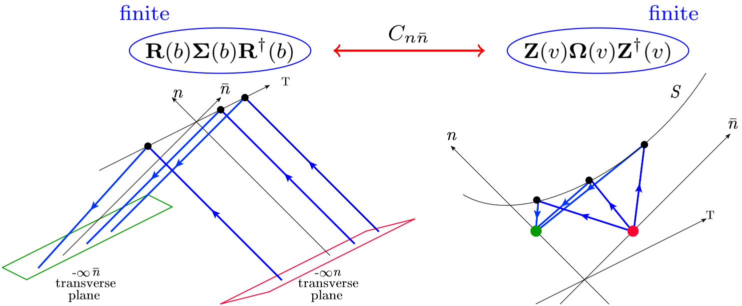
$$\mathbf{R}(b)\boldsymbol{\Sigma}(b)\mathbf{R}^\dagger(b) \quad \overset{C_{n\bar n}}{\longleftrightarrow} \quad \mathbf{Z}(v)\boldsymbol{\Omega}(v)\mathbf{Z}^\dagger(v)$$





In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization
- There are also UV renormalization factors in cusps (we omit them for a moment)



RDRT in conformal theory

In a conformal field theory rapidity divergences can be removed (*renormalized*) by a multiplicative factor.

$$C_{n\bar{n}}^{-1}\left(\mathbf{Z}(\{v\},\mu)\right) = \mathbf{R}_n(\{b\},\nu^+)$$

Rapidity anomalous dimension (RAD)

$$\mathbf{D}(\{b\}) = \frac{1}{2}\mathbf{R}_n^{-1}(\{b\},\nu^+)\nu^+\frac{d}{d\nu^+}\mathbf{R}_n(\{b\},\nu^+),$$

In CSS notation it is $-K$, in [Becher,Neubert] $F_{q\bar{q}}$, in SCET literature γ_ν.

(In CFT) DY-like Soft factors expresses as

$$\mathbf{\Sigma}(\{b\},\delta^+,\delta^-) \quad = \quad e^{2\mathbf{D}(\{b\})\ln(\delta^+/\nu^+)}\overbrace{\mathbf{\Sigma}_0(\{b\},\nu^2)}^{\text{finite}}e^{2\mathbf{D}^\dagger(\{b\})\ln(\delta^-/\nu^-)},$$

From conformal theory to QCD

QCD at the critical point

QCD is conformal in $4-2\epsilon^*$ dimensions

$$\beta(\epsilon^*)=0, \qquad \Rightarrow \qquad \epsilon^*=-a_s\beta_0-a_s^2\beta_1-....$$

It is very useful trick, allows to restore "conformal-violating" terms, see e.g.[Braun,Manashov,1306.5644]

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Thus, at $4-2\epsilon^*$ dimensions, the rapidity renormalization theorem works.

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RTRD works at any finite order of QCD

Proof by induction

- Important input: Counting of rap.div. is independent on number of dimensions
- Important input: At 1-loop QCD is conformal = RTRD hold.
- (1) All Leading divergences cancel by R.
- (2) Make shift $\epsilon^* \to \epsilon^* + \beta_0 a_s$.
- (3) Modify R such that next-to-leading divegences cancel (it can be done perturbatively, thanks to a_s)
- Repeat (2-3) N times, and got renormalization at a_s^{N+1} order.

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Soft factor has the form

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$$\mathbf{D}_{\text{QCD}} \neq \mathbf{D}_{\text{CFT}}$$

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Example then it does not work (no factorization?)

There are talks about "dipole-like" TMD distributions that could appear in processes like $pp \to hX$ e.g. [Boer,et al,1607.01654]

However, it is straightforward to show that the factorization is necessarily broken (or has not a closed form)

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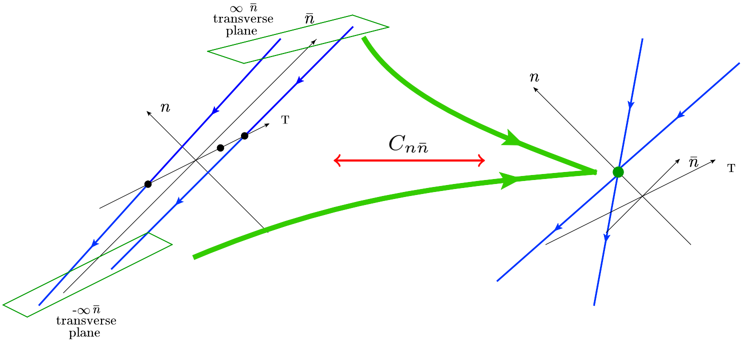

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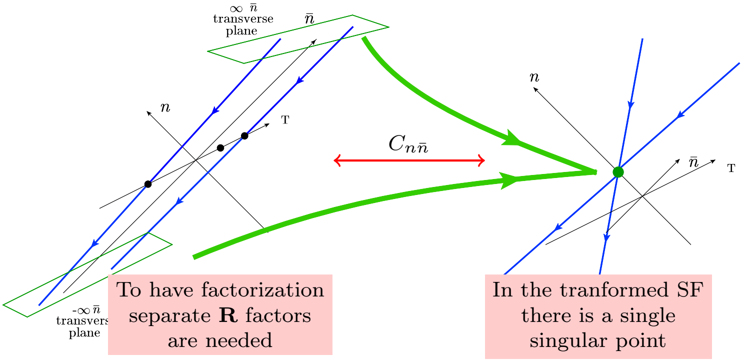

- The renormalization of dipole recouple colors → extra gauge link → ala BK equation.

Consequences

- Factorization for multi-Drell-Yan process (and TMD factorization as a particular case)
- Generalized (matrix) CSS equation
- Correspondence between soft and rapidity anomalous dimensions
- Constraints of soft anomalous dimension.
- Equality of DY and SIDIS TMD soft factors (?)
- Many others ... (*in progress*)

Part II:
Correspondence between
soft anomalous dimension (SAD)
&
rapidity anomalous dimension (RAD)

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Soft anomalous dimension (evolution of jet-production)

Scattering amplitude for n-massless partons (jets) at *fixed angles*

$$\mathcal{A}_n(\{v_i\}) \simeq \mathbf{H}_n(\{v_i\},\mu)\prod_{i=1}^{n}\frac{J(p_i,\mu)}{\mathcal{J}(v_i,\mu)}$$

$$\frac{d\mathbf{H}_n(\{v_i\},\mu)}{d\ln\mu} = \boldsymbol{\gamma}_s(\{v_i\})\times\mathbf{H}_n(\{v_i\},\mu).$$

$\boldsymbol{\gamma}_s(\{v_i\})$ is SAD.

Rapidity anomalous dimension (evolution of multiPDs)

The rapidity-divergences renomalized multiPD defined

$$F_f(\{x\},\{b\},\nu^+) = \mathbf{\Sigma}_0(\{b\},\nu^2)\mathbf{R}^{\dagger-1}(\{b\},\nu^-)\tilde{F}_f(\{x\},\{b\})$$

$$\nu^+\frac{d}{d\nu^+}F(\{x\},\{b\},\mu,\nu^+) = \frac{1}{2}\mathbf{D}(\{b\},\mu)\times F(\{x\},\{b\},\mu,\nu^+).$$

$\mathbf{D}(\{b\},\mu)$ is RAD.

Soft/rapidity anomalous dimension correspondence

The equivalence (under conformal transformation) between **Z** and **R** implies the equality between corresponding anomalous dimensions

$$\gamma_s(\{v\}) = 2\mathbf{D}(\{b\})$$

It has been observed in $\mathcal{N} = 4$ SYM [Li,Zhu,1604.01404].

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

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- At ϵ^* conformal symmetry of QCD is restored

In QCD

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- Exact relation!
- Connects different regimes of QCD

→ Lets test it.

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

How to use it?

- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$
- $\epsilon^* = 0 - a_s\beta_0 - a_s^2\beta_1 - a_s^3\beta_2 - ....$
- We can compare order by order in PT

$$\begin{aligned}
\mathbf{D}_1(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_1(\{v\}),\\
\mathbf{D}_2(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_2(\{v\}) + \beta_0\mathbf{D}_1'(\{b\}),\\
\mathbf{D}_3(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_3(\{v\}) + \beta_0\mathbf{D}_2'(\{b\}) + \beta_1\mathbf{D}_1'(\{b\}) - \frac{\beta_0^2}{2}\mathbf{D}_1''(\{b\}),
\end{aligned}$$

TMD rapidity anomalous dimension

2-loop expression for RAD

$$\mathcal{D}_1(\mathbf{b}^2,\epsilon) = -2a_sC_F\Big[\Big(\frac{\mathbf{b}^2}{4}\Big)^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\Big] = a_sC_F\Big\{2\mathbf{L}_\mu+\epsilon\underbrace{(\mathbf{L}_\mu^2+\zeta_2)}_{D_1'}+...\Big\}$$

Taking

$$\gamma_s = C_Fa_s(\Gamma_0\mathcal{L}_\mu-\tilde{\gamma}_0)+C_Fa_s^2(\Gamma_1\mathcal{L}_\mu-\tilde{\gamma}_1)+... \tag{1}$$

We find

$$\mathcal{D}_2(\mathbf{b}^2,0) = C_F\Big(\beta_0\mathbf{L}_\mu^2+\frac{\Gamma_1}{2}\mathbf{L}_\mu\underbrace{-\frac{\tilde{\gamma}_1}{2}+\beta_0\zeta_2}_{d^{(2,0)}}\Big) \tag{2}$$

It coincides with the direct calculation [Echevarria,Scimemi,AV,1511.05590].

TMD rapidity anomalous dimension

3-loop expression for RAD

$$\mathcal{D}_2(\mathbf{b}^2,\epsilon) = a_s^2 C_F\Big\{\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E) + \frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\Big(\frac{3(4-3\epsilon)}{2\epsilon}C_A - N_f\Big)\Big) + \boldsymbol{B}^{\epsilon}\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0 + \frac{\beta_0}{2\epsilon^2} - \frac{\Gamma_1}{2\epsilon}\Big\}$$

Taking

$$\gamma_s = C_F a_s(\Gamma_0\mathcal{L}_\mu - \tilde\gamma_0) + C_F a_s^2(\Gamma_1\mathcal{L}_\mu - \tilde\gamma_1) + C_F a_s^3(\Gamma_2\mathcal{L}_\mu - \tilde\gamma_2) + ...$$

We find

$$\mathcal{D}_3(\mathbf{b}^2,0) = \text{logs} - \frac{\tilde\gamma_2}{2} + (\beta_1+\beta_0\Gamma_1)\zeta_2 - \frac{2}{3}\beta_0^2\zeta_3 + \beta_0\Big\{C_A\Big(\frac{2428}{81} - 26\zeta_4\Big) - N_f\frac{328}{81}\Big\}$$

It coincides with the direct calculation [Li,Zhu,1604.01404].

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$$\mathcal{D}^{(3)}_{L=0} = -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3 - \frac{88}{3}\zeta_2\zeta_3 - 192\zeta_5 - \frac{297029}{729} + \frac{6392}{81}\zeta_2 + \frac{154}{3}\zeta_4\right)$$
$$-\frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3 + \frac{62626}{729} - \frac{824}{81}\zeta_2 + \frac{20}{3}\zeta_4\right) -$$
$$\frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3 + \frac{1711}{27} - 16\zeta_4\right) - \frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3 - \frac{1856}{729}\right)$$

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Quadrupole part of SAD

$$\boldsymbol{\gamma}_s(\{v\}) = -\frac{1}{2}\sum_{[i,j]} \mathbf{T}_i^A \mathbf{T}_j^A \gamma_{\text{dipole}}(v_i \cdot v_j) - \sum_{[i,j,k,l]} if^{ACE} if^{EBD} \mathbf{T}_i^A \mathbf{T}_j^B \mathbf{T}_k^C \mathbf{T}_l^D \mathcal{F}_{ijkl} - \sum_{[i,j,k]} \mathbf{T}_i^{\{AB\}} \mathbf{T}_j^C \mathbf{T}_k^D if^{ACE} if^{EBD} C + \mathcal{O}(a_s^4),$$

Quadrupole part has been calculated in [Almelid,Duhr,Gardi;1507.00047]

$$\tilde{C} = a_s^3\left(\zeta_2\zeta_3 + \frac{\zeta_5}{2}\right) + \mathcal{O}(a_s^4),$$

$$\tilde{\mathcal{F}}_{ijkl}(\{b\}) = 8a_s^3 \mathcal{F}(\tilde{\rho}_{ikjl}, \tilde{\rho}_{iljk}) + \mathcal{O}(a_s^4),$$

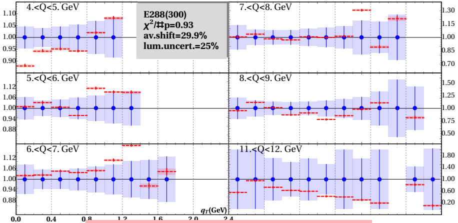
Quadrupole part of RAD

- Color structures are not affected by ϵ^*
- Quadrupole contribution depends only on conformal ratios

$$\rho_{ijkl} = \frac{(v_i \cdot v_j)(v_k \cdot v_l)}{(v_i \cdot v_k)(v_j \cdot v_l)} \quad \leftrightarrow \quad \tilde{\rho}_{ijkl} = \frac{(b_i - b_j)^2 (b_k - b_l)^2}{(b_i - b_k)^2 (b_j - b_l)^2}$$

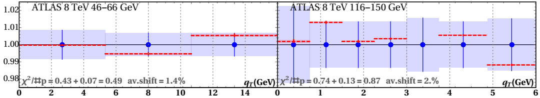
Conclusion

Global extraction of F_1 and RAD at NNLO by [Bertone, Scimemi & AV,1902.????]



Drell-Yan at $Q = 4 - 5$GeV

Drell-Yan at $Q = 116 - 150$GeV



TMD evolution is a key element

$$\frac{\chi^2_{\text{global}}}{d.o.f.} \simeq 1.15$$

Here:

- 3-loop evolution
- 2-loop coefficient function
- 2-loop matching
- **ζ-prescription**

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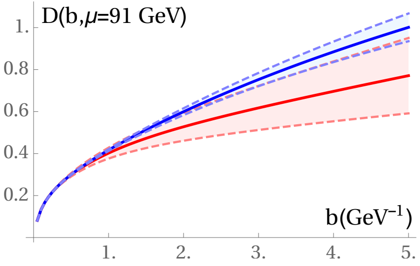
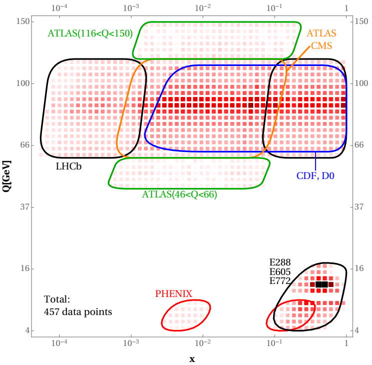
Non-perturbative part of RAD

At large-**b** RAD became non-perturbative and must be extracted from the data together with TMDs.





Conclusion

Renormalization theorem for rapidity divergences

- Rapidity divergences in DY-like soft factors can be renomalized (just like UV divergences).
- Conformal-stereographic projection provides simple criterion of renormalizability.
- It results to a consistent definition of TMD distributions, DPDs, multi-PDs.

Further progress

- TMD evolution (double-scale evolution) and ζ-prescription [Scimemi,AV,1803.11089]
- Rapidity divergences in operators [Scimemi,Tarasov,AV,1901.](sec.4.2)
- Non-perturbative definition of RAD [Schaefer,AV,in progress]

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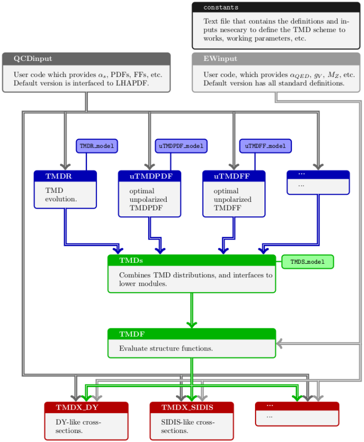
arTeMiDe v1.4



- Variety of evolutions
- LO, NLO, NNLO
- No restriction for NP models
- Fast FORTRAN code + python-interface (under development)
- DY cross-sections
- SIDIS cross-sections (not tuned yet)
- Theory uncertainty bands

https://github.com/VladimirovAlexey/artemide-public

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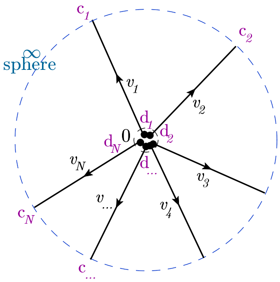
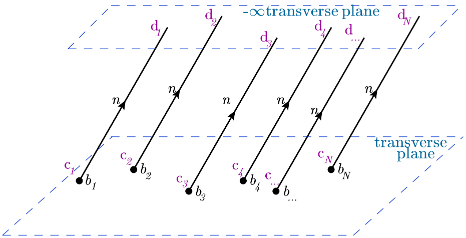
Conformal-stereographic transformation

$$\mathcal{C}_{\bar{n}} : \{x^+, x^-, x_\perp\} \to \{\frac{-1}{2a}\frac{1}{\lambda + 2ax^+}, x^- + \frac{ax_\perp^2}{\lambda + 2ax^+}, \frac{x_\perp}{\lambda + 2ax^+}\}$$

- Translation – special conformal transformation (along n) – Translation
- a and λ are free parameters





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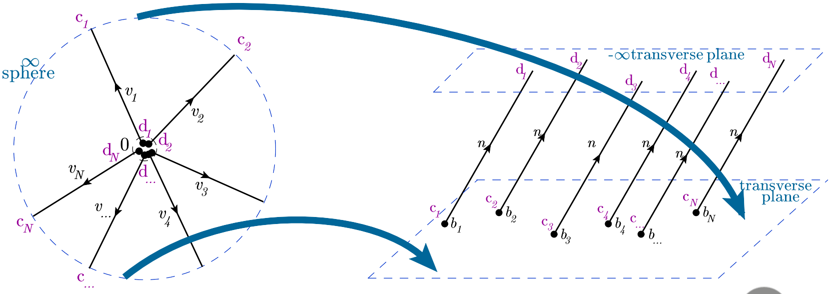

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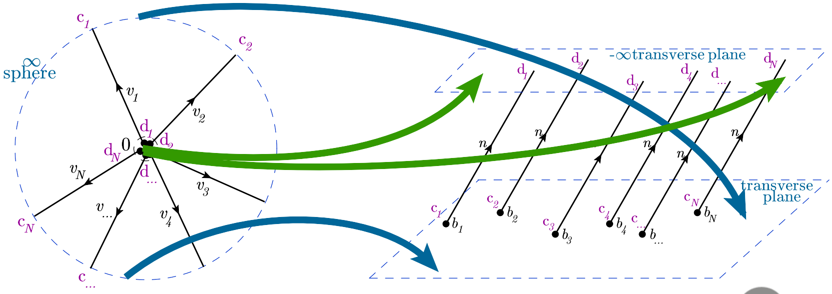

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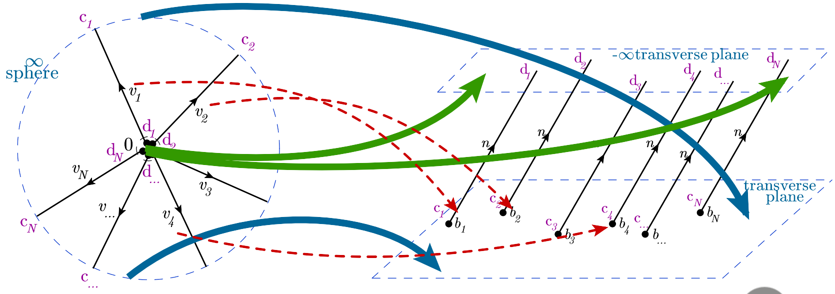

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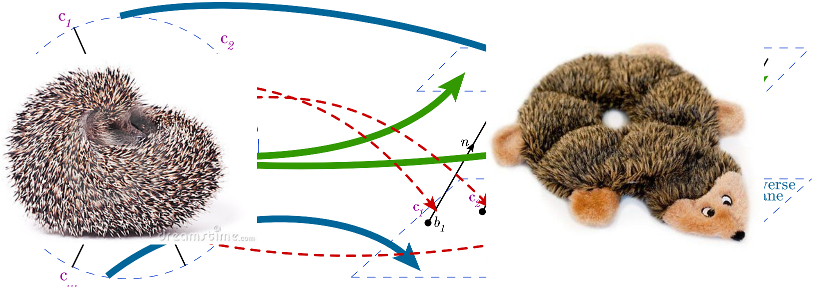

TMD evolution equations

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (3)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (4)$$

Solution: $F(x,\mathbf{b};\mu_f,\zeta_f) = R[\mathbf{b};(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,\mathbf{b};\mu_i,\zeta_i)$

- γ_F – TMD anomalous dimension
- $\mathcal{D}$ – rapidity anomalous dimension ($= -\frac{\tilde{K}}{2}$[Collins' book], $= K$[Bacchetta, at al,1703.10157])
- Anomalous dimensions are *universal*, i.e. depend only on flavor (gluon/quark).

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TMD evolution is two-dimensional

$$\begin{pmatrix} \mu^2 \dfrac{d}{d\mu^2} \\ \zeta \dfrac{d}{d\zeta} \end{pmatrix} F = \begin{pmatrix} \dfrac{\gamma_F}{2} \\ -\mathcal{D} \end{pmatrix} F$$

⟷

$$\overrightarrow{\boldsymbol{\nabla}} F = \overrightarrow{\mathbf{E}} F$$

$\overrightarrow{\mathbf{E}}$ is 2D evolution field in $\overrightarrow{\boldsymbol{\nu}} = (\ln \mu^2, \ln \zeta)$ coordinates

TMD evolution is two-dimensional

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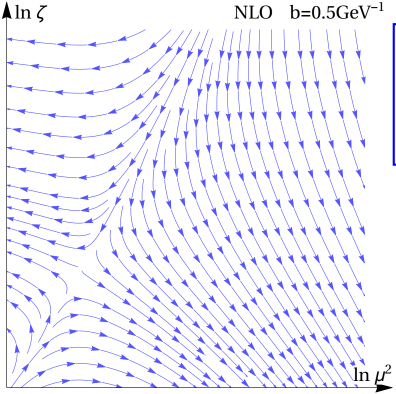

Solution

$$R[(\mu_f, \zeta_f) \to (\mu_i, \zeta_i)] = \exp\left(\int_P d\vec{\boldsymbol{\nu}} \cdot \vec{\mathbf{E}}\right)$$

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↔

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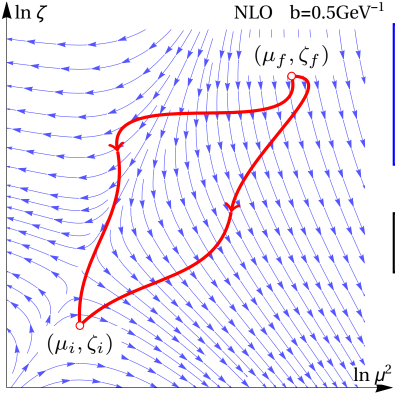

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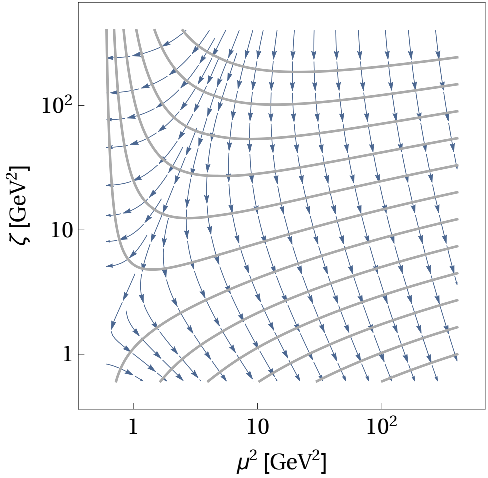
The integration path is unimportant!

TMD distribution is not defined by a scale (μ, ζ)
It is defined by an equipotential line.



The scaling is defined by
~~a difference between scales~~
a difference between potentials

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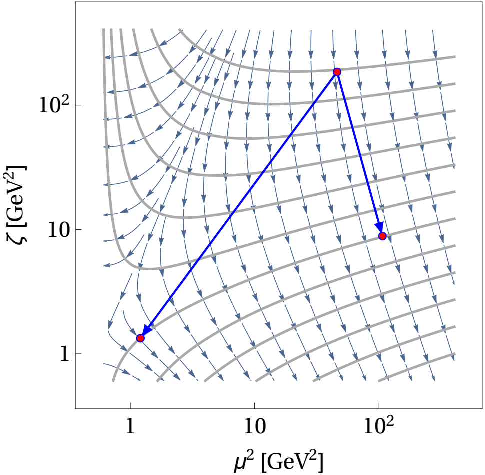
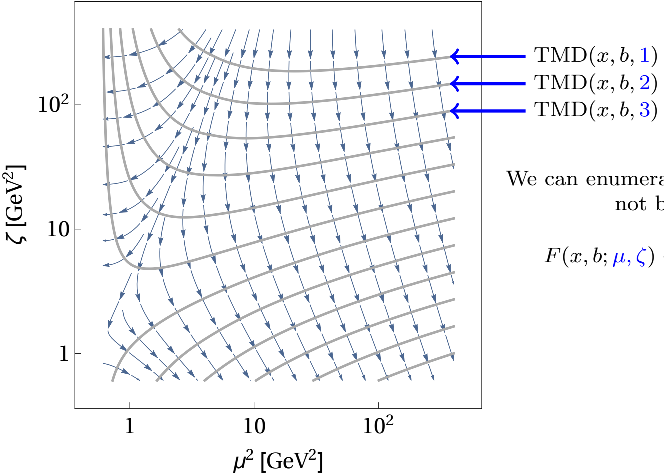


The scaling is defined by
~~a difference between scales~~
a difference between potentials

Evolution factor to both points
is the same
although the scales are
different by 10^2GeV^2

TMD distributions on the same equipotential line are equivalent.



We can enumerate them by a lines not by (μ, ζ)

$$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$$

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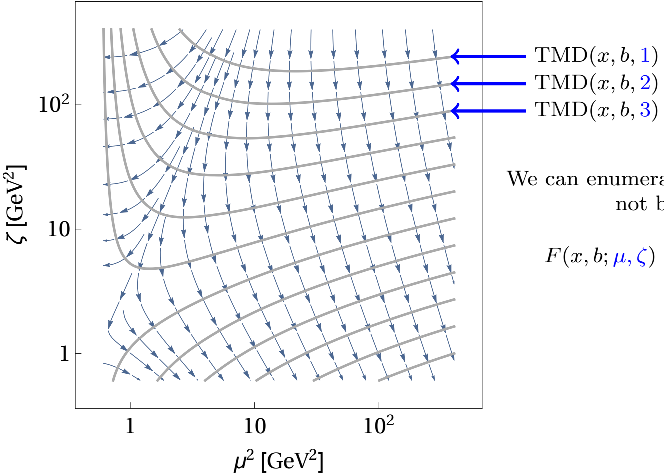

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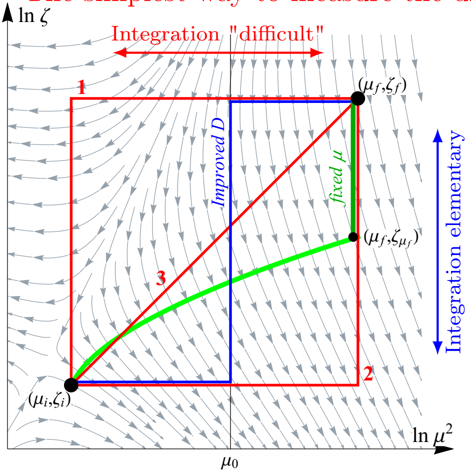
$$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$$

The simplest way to measure the difference between potentials



$$R = \left(\frac{\zeta_f}{\zeta_{\mu_f}}\right)^{-\mathcal{D}(\mu_f,b)}$$

- Numerically simple (and fast). Compare to

$$\times \exp\left\{\ln\frac{\sqrt{\zeta_A}}{\mu_b}\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_D(g(\mu');1) - \ln\frac{\sqrt{\zeta_A}}{\mu'}\gamma_K(g(\mu'))\right]\right\}. \quad (13.70$$

- $\mu_f = Q$ thus a_s is small
- **It is different representation of the Sudakov exponent.**

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