TMD evolution
as
a double-scale evolution

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based on [1803.11089]

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Evolution of transverse momentum dependent (TMD) distributions

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TMD evolution

Disclaimer

The content of the talk is almost trivial
However, it appears difficult to describe in a well-prepared auditory,
because despite the long history of TMD evolution people never analyzed it seriously.

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Motivation

TMD factorization describes p_T-spectrum of the "double-inclusive processes", at the range of small-p_T, where the transverse momentum is dominantly generated by the intrinsic motion of partons.

Current status

The study of TMD distribution enters the new phase: precise extraction of TMD distributions.

Experiment

- Large amount of Drell-Yan data: from 5GeV to 120 GeV
- Some data is ultimately precise (ATLAS)
- Large amount of SIDIS data (low energy only)
- e^+e^--annihilation data (not well studied yet)

Theory

- Each part of TMD factorization is proved
- Perturbative parts are known up high orders
  - Hard part: 3-loops
  - Evolution: 3-loops
  - Matching: 2-loop (1-loop polarized)

TMD phenomenology

- Many separate fits of subsets of data.
- (practically) All studies are done at LO (often without evolution).
- There is the first global fit of DY+SIDIS [Bacchetta, et al; 1703.10157].
- There is a single example of higher perturbative order fit (NLO, NNLO) [Scimemi,AV; 1706.01473].
- There is only as single attempt to estimate theory uncertainty band [Scimemi,AV; 1706.01473].

Current global goal

To join theory achievements and to apply it the modern (precise) data, and make the global determination of TMD functions.

Questions of internal consistency and comparison of results
are ultimately important.

Outline

Disclaimer:
part of talk is probably too elementary. But I have not found any dedicated discussion in the literature.
Thus, it is probably worth to spend some time and fix the terminology.

Outline

- TMD evolution in a nutshell
  - Equations, solutions, etc.
  - TMD evolution field and its structure
- Effects of truncation perturbation theory
  - Violation of integrability condition, and solution-dependence of TMD evolution
  - Methods to fix the ambiguity.
- ζ-prescription
  - Physical meaning of ζ-prescription
  - Universal scale-independent TMD distribution.
- TMD cross-section and perturbative uncertainties.

TMD evolution is used for two practical purposes

- Compare different experiments
- Modeling TMD distribution

$$\frac{d\sigma}{dX} \sim \int d^2b\, e^{i(bq_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1,b;\mu,\zeta_1) F_{f'\leftarrow h}(x_2,b;\mu,\zeta_2)$$

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Minimize $\ln(Q/\mu)$
$\mu = Q$

$\zeta_1\zeta_2 = Q^4$
or
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Minimize $\mathbf{L}_\mu$, $\mathbf{L}_{\sqrt{\zeta}}$
$\mu \sim \sqrt{\zeta} \sim b^{-1}$

$$F(x,b;\mu,\zeta) \sim C(x,b;\mu,\zeta) \otimes \mathrm{PDF}(x,\mu)$$

Typical model for TMD includes matching

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Minimize $\ln(Q/\mu)$
$\mu = Q$

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$$F(x,b;\mu_f,\zeta_f) = R[b,(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,b;\mu_i,\zeta_i)$$

Final scale → TMD evolution factor → Initial scale

Minimize $\mathbf{L}_\mu$, $\mathbf{L}_{\sqrt{\zeta}}$
$\mu \sim \sqrt{\zeta} \sim b^{-1}$

$$F(x,b;\mu,\zeta) \sim C(x,b;\mu,\zeta) \otimes \mathrm{PDF}(x,\mu)$$

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TMD evolution: theory

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

- γ_F – TMD anomalous dimension
- $\mathcal{D}$ – rapidity anomalous dimension ($= -\frac{\tilde{K}}{2}$[Collins' book], $= K$[Bacchetta, at al,1703.10157])
- Anomalous dimensions are *universal*, i.e. independent on hadron, polarization, PDF/FF(see proof [AV;1707.07606]).
- Anomalous dimension depend only on flavor (gluon/quark). Skip index f in the following.

TMD evolution: theory

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

Solution: $F(x,\mathbf{b};\mu_f,\zeta_f) = R[\mathbf{b};(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,\mathbf{b};\mu_i,\zeta_i)$

Expression for R is known as "Sudakov exponent"

$$\times \exp\left\{ \ln\frac{\sqrt{\zeta_A}}{\mu_b}\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_D(g(\mu');1) - \ln\frac{\sqrt{\zeta_A}}{\mu'}\gamma_K(g(\mu'))\right]\right\}. \quad (13.70)$$

This is probably the best formula for calculating and fitting TMD fragmentation functions;

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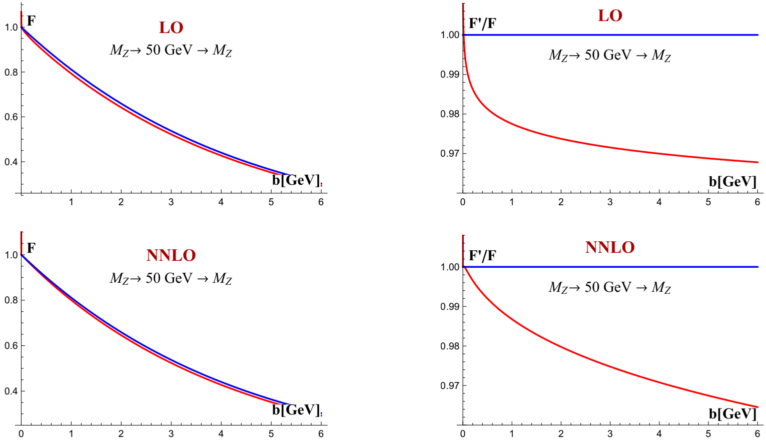
TMD evolution is a cornerstone of the construct, and it is not as simple as it looks.

- There are theoretical traps in TMD evolution, and I have not found a single discussion on it.
- Each problem is small, but there are many of them.
- They became evident at high-perturbative order.

Problem 1: Violation of transitivity

$$R[\mathbf{b}; X \to Y]\ R[\mathbf{b}; Y \to X] = 1$$



There is a violation of transitivity ~ 2 % which seems better at NNLO

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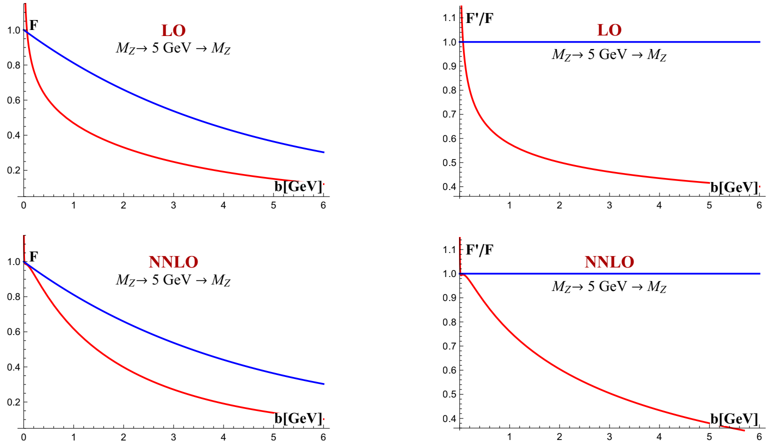
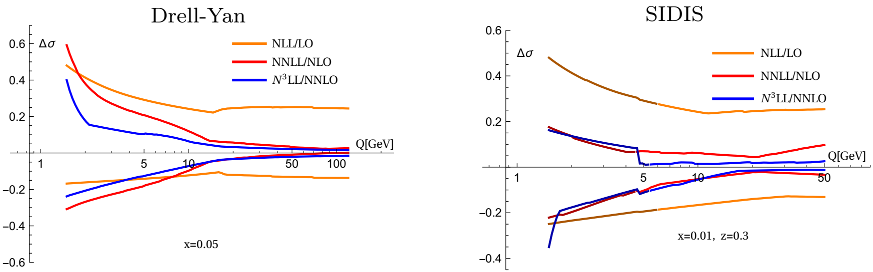


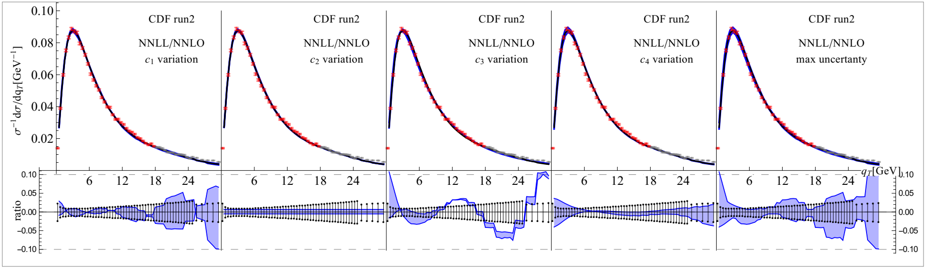
There is a VERY strong violation of transitivity, which seems worse at NNLO

Problem 2: Anomalous behavior of variations bands



- The variations of constants does not decrease at large-Q.
- Opposite it start to increase at large-Q.
- NNLO band seems larger then NLO

Problem 3: Anomalous behavior of variations



In [Scimemi,AV; 1706.01473] there was a study of a perturbative stability. With the help of variation of scales.

- The variations of constants c_1 and c_3 are the largest despite these are 3-loop series (compare to c_2 and c_4 which are 2-loop)
- The variation of c_1 and c_3 are numerically unstable (see artifacts)

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Problem 4: Strong dependence on μ

- It seems that TMD fits are seriously dependent on the values of μ (μ_b, μ^*, etc)
- Often the parameter μ is used as a subject of fit.
- Is it evidence of perturbative instability? Difficult to answer, since there is no dedicated study on it.

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- It seems that TMD fits are seriously dependent on the values of μ (μ_b, μ^*, etc)
- Often the parameter μ is used as a subject of fit.
- Is it evidence of perturbative instability? Difficult to answer, since there is no dedicated study on it.

In fact, these are consequences of a larger problem:

> not self-consistency of TMD evolution in the "traditional" form within perturbation theory.

Under "traditional" I refer to, say [Collins textbook],[Aybat,Rogers,1101.5057],[Echevarria,et al,1208.1281],...

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Let us examine the TMD evolution equation again

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta),$$
$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta),$$

The solution of TMD evolution equation (i.e. R) exists (in the mathematical sense) only if

$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} = -\mu^2 \frac{d}{d\mu^2} \mathcal{D}(\mu,b)$$

integrability condition

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integrability condition

Solution is

$$R[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)] = \exp\left[\int_P \left(\gamma_F(\mu,\zeta)\frac{d\mu}{\mu} - \mathcal{D}(\mu,b)\frac{d\zeta}{\zeta}\right)\right]$$

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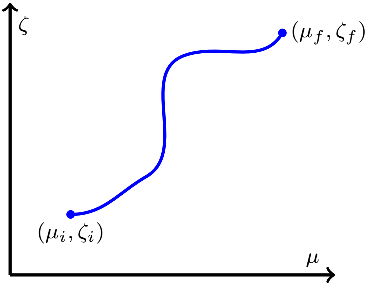
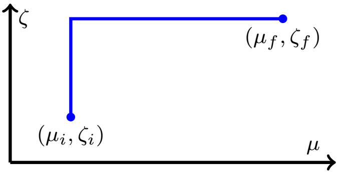


The solution is independent on the path of the integration due to *integrability condition*

Examples



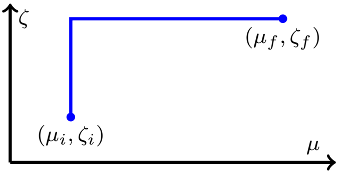
Solution 1

$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - \mathcal{D}(\mu_i, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$

[Collins' textbook],[Aybat,Rogers,1101.5057],...

99% popular

Examples



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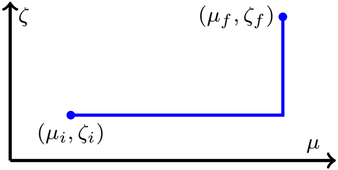
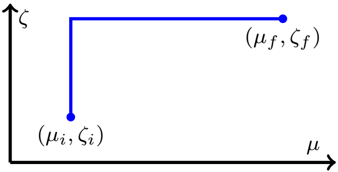


Solution 2

$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - \mathcal{D}(\mu_f, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$

Examples



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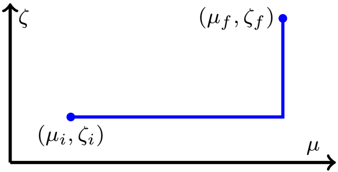

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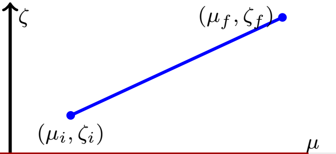

Solution 3

$$\ln R = \int_0^1 \Big(\gamma_F(\mu(t),\zeta(t))\frac{\mu_f-\mu_i}{(\mu_f-\mu_i)t+\mu_i} - \mathcal{D}(\mu(t),b)\frac{\zeta_f-\zeta_i}{(\zeta_f-\zeta_i)t+\zeta_i}\Big)dt$$

TMD evolution is essentially 2D task.
Let me introduce convenient notation.

Evolution scales	Anomalous dimensions
$$\boldsymbol{\nu} = (\ln\left(\frac{\mu^2}{1\ \text{GeV}^2}\right), \ln\left(\frac{\zeta}{1\ \text{GeV}^2}\right)).$$	$$\mathbf{E}(\boldsymbol{\nu}, b) = (\frac{\gamma_F(\boldsymbol{\nu})}{2}, -\mathcal{D}(\boldsymbol{\nu}, b)).$$
2d vector	vector field

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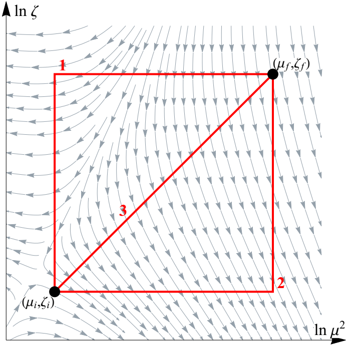

Evolution equation

$$\boldsymbol{\nabla} F(x, b; \boldsymbol{\nu}) = \mathbf{E}(\boldsymbol{\nu}, b) F(x, b; \boldsymbol{\nu})$$

Solution

$$\ln R[b, \boldsymbol{\nu}_f \to \boldsymbol{\nu}_i] = \int_P \mathbf{E} \cdot d\boldsymbol{\nu}$$

Scalar potential

The integrability condition is the condition that evolution field **E** is *irrotational* (*conservative*)

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

Thus, it is determined by a *scalar potential*

$$\mathbf{E}(\boldsymbol{\nu}, b) = \boldsymbol{\nabla} U(\boldsymbol{\nu}, b)$$

Scalar potential

The integrability condition is the condition that evolution field $\mathbf{E}$ is *irrotational* (*conservative*)

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Evolution is the difference between potentials

$$\ln R[b; \boldsymbol{\nu}_f \to \boldsymbol{\nu}_i] = U(\boldsymbol{\nu}_f, b) - U(\boldsymbol{\nu}_i, b).$$

Scalar potential can be easily found

$$U(\boldsymbol{\nu}, b) = \int^{\nu_1} \frac{\Gamma(s)s - \gamma_V(s)}{2} ds - \mathcal{D}(\boldsymbol{\nu}, b)\nu_2 + \text{const}(b),$$

TMD evolution in the perturbation theory

In the real live we can operate only with the first several terms of perturbation theory. Therefore, the *integrability* condition is violated

$$\mu\frac{d\mathcal{D}(\mu)}{d\mu}\neq\Gamma(\mu)$$

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Simple example at 1-loop

$$\mathcal{D}=a_s(\mu)\frac{\Gamma_0}{2}\mathbf{L}_\mu$$

$$\begin{aligned}\mu\frac{d\mathcal{D}}{d\mu} &= a_s(\mu)\frac{\Gamma_0}{2}\left(\mu\frac{d}{d\mu}\mathbf{L}_\mu\right)+\left(\mu\frac{da_s(\mu)}{d\mu}\right)\frac{\Gamma_0}{2}\mathbf{L}_\mu \\ &= a_s(\mu)\Gamma_0-\beta_0a_s^2(\mu)\Gamma_0\mathbf{L}_\mu\neq a_s(\mu)\Gamma_0\end{aligned}$$

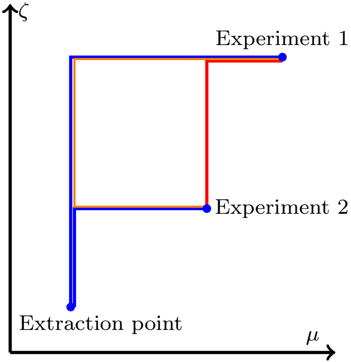
At N'th order of perturbation theory $\Gamma-d\mathcal{D}\sim a_s^{N+1}\mathbf{L}_\mu^N$

- Since $a_s\sim\ln^{-1}\mu$ there is always (at any finite N) value of b(fixed) then $\delta\Gamma\gg1$
- The value of μ does not play a role
- In fact, this term is ALWAYS NLO, in the standard resummation counting ($a_sL\sim1$).
- The NP models for $\mathcal{D}$ only enforce the problem.

In PT the TMD evolution is dependent on the path

Transitivity

$$R[b;(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)] = R[b;(\mu_1,\zeta_1)\to(\mu_3,\zeta_3)]R[b;(\mu_3,\zeta_3)\to(\mu_2,\zeta_2)]$$

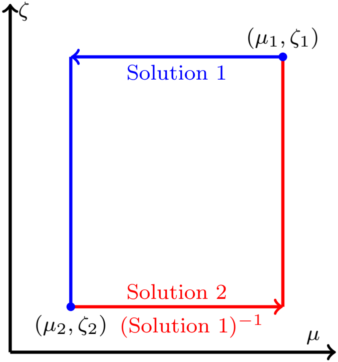


In the fitting procedure different experiments (different Q) define the same point (same b)

But there is simple connection between different Q's

Inversion

$$R[b;\{\mu_1,\zeta_1\}\to\{\mu_2,\zeta_2\}] = R^{-1}[b;\{\mu_2,\zeta_2\}\to\{\mu_1,\zeta_1\}]$$



$$R[b;\{\mu_1,\zeta_1\}\xrightarrow{1}\{\mu_2,\zeta_2\}] = R^{-1}[b;\{\mu_2,\zeta_2\}\xrightarrow{2}\{\mu_1,\zeta_1\}]$$
$$\neq R^{-1}[b;\{\mu_2,\zeta_2\}\xrightarrow{1}\{\mu_1,\zeta_1\}]$$

There is no simple way to compare different fits!

Reverse ingeneering in each case!

Helmeholz decomposition

$$\mathbf{E} = \tilde{\mathbf{E}} + \boldsymbol{\Theta}$$

$\tilde{\mathbf{E}}$	conservative (*irrotational*) component	$\text{curl}\tilde{\mathbf{E}} = 0$
$\boldsymbol{\Theta}$	*divergence-free* component	$\boldsymbol{\nabla} \cdot \boldsymbol{\Theta} = 0$

$$\tilde{\mathbf{E}} \cdot \boldsymbol{\Theta} = 0$$

$$\text{curl}\mathbf{E} = \text{curl}\boldsymbol{\Theta} = \frac{\delta\Gamma}{2}$$

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Ambiguous scalar potential

The *divergence-free* component is an artifact of truncated PT. It prevents the definition of scalar potential

$$\boldsymbol{\nabla}\tilde{U} = \tilde{\mathbf{E}}, \qquad \mathbf{curl}V = \boldsymbol{\Theta}$$

$$\nabla^2\tilde{U} = \frac{d\gamma_F}{d\ln\mu} \qquad \text{vs.} \qquad \boldsymbol{\nabla}U = \mathbf{E}$$

Poisson equation solution is defined up to $\nabla^2 f = 0$.

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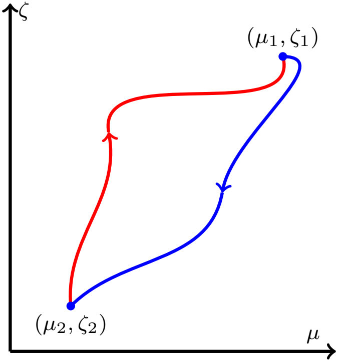
Non-conservative evolution



$$\oint_C \mathbf{E} \cdot d\nu = \int_\Omega d^2\nu \, \mathrm{curl}\boldsymbol{\Theta} = \frac{1}{2} \int_\Omega d^2\nu \, \delta\Gamma(\nu, b)$$

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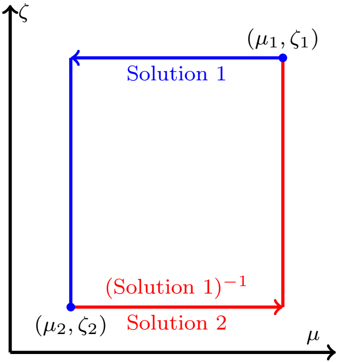
Non-conservative evolution



$$\oint_C \mathbf{E}\cdot d\nu = \int_\Omega d^2\nu \,\mathrm{curl}\boldsymbol{\Theta} = \frac{1}{2}\int_\Omega d^2\nu\,\delta\Gamma(\nu,b)$$

$$\ln\frac{\text{solution 1}}{\text{solution 2}} = \ln\left(\frac{\zeta_f}{\zeta_i}\right)\int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\delta\Gamma(\mu,b)$$

The "longer" evolution – the bigger error

That is why for $Z-$boson error is larger

How to fix it?

There are many possibilities

- Lets use a single evolution line $\mu^2 = \zeta$, and the solution 3
  - + Restore self-consistency and inversion
  - - - Everyone stick to a single line. No freedom for modeling.
  - - Numerically more expensive
- Lets set $\boldsymbol{\Theta} = 0$, and use only $\tilde{\mathbf{E}}$
  - + + Ideal solution which does not restrict anything
  - - The procedure is not unique, we need to set boundary conditions
- Lets repair the integrability condition by adding terms beyond PT
  - + + Very simple
  - - The procedure is not unique (however, there is only single "good" solution)
  - Equivalent to some boundary condition (do not know which)

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In PT the integrability condition is violated

We can repair it by accounting "higher-then-allowed" terms of perturbation theory

$$\mu\frac{d\mathcal{D}(\mu,b)}{d\mu} \neq -\zeta\frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{D}(\mu,b)\neq\Gamma(\mu)$$

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Improved $\mathcal{D}$ scenario

$$\mu\frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta\frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu)$$

$$\mathcal{D}(\mu,b) = \int_{\mu_0}^{\mu}\frac{d\mu}{\mu}\Gamma(\mu) + \mathcal{D}(\mu_0,b) \qquad \text{[CS,1981]}$$

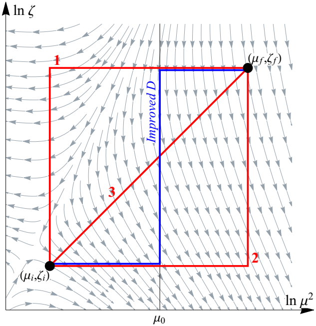
$$\ln R[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i);\mu_0] = \int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\left(\Gamma(\mu)\ln\left(\frac{\mu^2}{\zeta_f}\right) - \gamma_V(\mu)\right) - \int_{\mu_0}^{\mu_i}\frac{d\mu}{\mu}\Gamma(\mu)\ln\left(\frac{\zeta_f}{\zeta_i}\right) - \mathcal{D}(\mu_0,b)\ln\left(\frac{\zeta_f}{\zeta_i}\right).$$

- μ_0 is some new scale where "perturbation theory works".
- In fact it is the composition of solution 1 and 2
- Also possible to use resummed expression (no μ_0, but limited range of b) [Echevarria,et al,1208.1281]

Improved $\mathcal{D}$ scenario

$$\mu\frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta\frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu)$$



- Transitivity and inversion hold
  If μ_0 is kept explicit (not $\mu_0 = \mu_i$ as typically used)
- If different μ_0 are used, the problem of comparison returns
- If different non-perturbative models are used, the problem also returns
- The evolution (quite strongly) depends on μ_0 (c_1 variation band)

Improved γ scenario
Use integrability condition as the definition

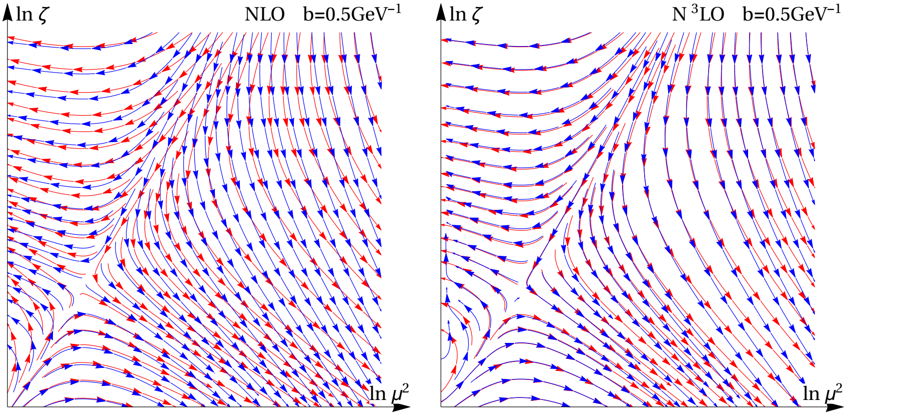
$$\mu \frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta \frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\gamma_F(\mu,\zeta) \to \gamma_M(\mu,\zeta,b) = -\mu\frac{d}{d\mu}\mathcal{D}(\mu,b)\ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_V(\mu)$$

$$\begin{aligned}\ln R[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)] &= -\int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\left(2\mathcal{D}(\mu,b)+\gamma_V(\mu)\right)\\ &+\mathcal{D}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right) - \mathcal{D}(\mu_i,b)\ln\left(\frac{\mu_i^2}{\zeta_i}\right).\end{aligned}$$

- Explicitly transitive, and inverse.
- Simple non-perturbative generalization ($\mathcal{D} \to \mathcal{D}_{NP}$)
- No extra scales. The evolution field is explicitly conservative.

How strong is modification of the field?



Part 2:
ζ-prescription

The final scales (μ_f, ζ_f) are fixed by process kinematics $\sim (Q, Q^2)$.
The initial scale are fixed only by model of TMD distribution.

Small-b matching

At small-b one can match TMD to collinear distribution by OPE

$$\text{TMD}(x, b; \mu_i, \zeta_i) = C(x, \mathbf{L}_\mu, \mathbf{L}_{\sqrt{\zeta}}, \mu) \otimes \text{PDF}(x, \mu)$$

- It is often used as an zero-level input to the model of TMD.
- It guaranties agreement with high energy experiments.
- It also requires the evolution from $(Q, Q^2) \to (\mu_i, \zeta_i)$, which are typically selected as

$$\mu_i^2 = \zeta_i \sim \frac{1}{b^2}$$

TMD case

$$d\sigma \sim \int d^2b e^{iqb} H(Q)\{R\Big(Q \to \frac{1}{b}\Big)\}^2 F_1(x_1, b; b^{-1}, b^{-2}) F_2(x_2, b; b^{-1}, b^{-2})$$

This is the standard approach that is used in majority of applications.

$$F_1(x_1, b; b^{-1}, b^{-2}) \to \text{phenomenological parametrization}$$

Analogy in DIS

$$d\sigma \sim C(Q,x) \otimes R\Big(Q \to 1/x\Big) \otimes f(x, 1/x)$$

$$f(x, 1/x) \to \text{phenomenological parametrization}$$

TMD case

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$$f(x, 1/x) \to \text{phenomenological parametrization}$$

It is complete non-sense!

TMD case

$$d\sigma \sim \int d^2b e^{iqb} H(Q)\{R\Big(Q \to \frac{1}{b}\Big)\}^2 F_1(x_1,b;b^{-1},b^{-2})F_2(x_2,b;b^{-1},b^{-2})$$

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Analogy in DIS

~~$d\sigma \sim C(Q,x) \otimes R\Big(x;Q \to 1/x\Big) \otimes f(x,1/x)$~~

$$d\sigma \sim C(Q,x) \otimes R\Big(x;Q \to 1\text{GeV}\Big) \otimes f(x,1\text{GeV})$$

$$f(x,1\text{GeV}) \to \text{phenomenological parametrization}$$

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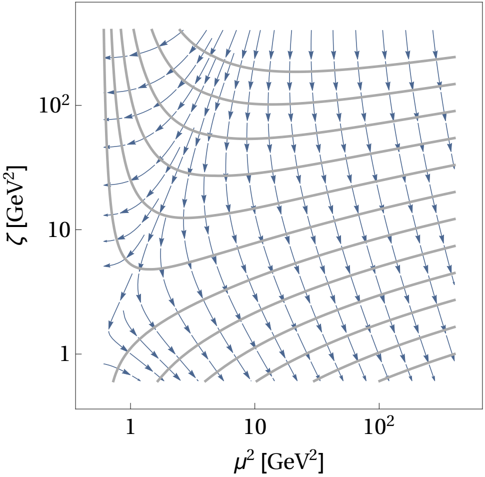
In the TMD case there is no notion of a scale, because it is defined on a plane



The scaling is defined by
~~a difference between scales~~
a difference between potentials

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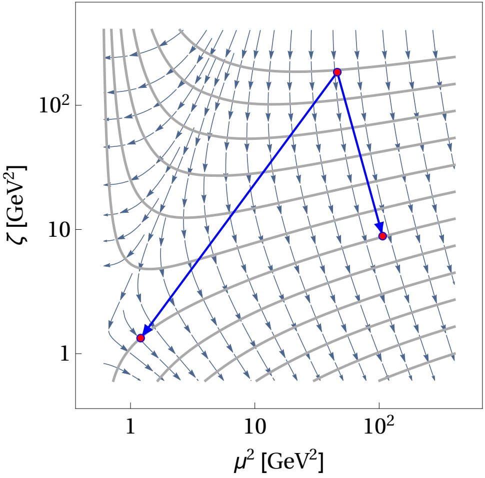
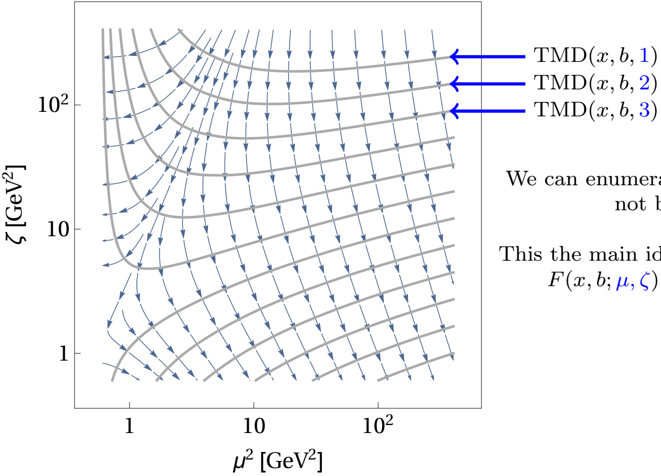


The scaling is defined by
~~a difference between scales~~
a difference between potentials

These evolution are the same although the scales are different by 10^2GeV^2

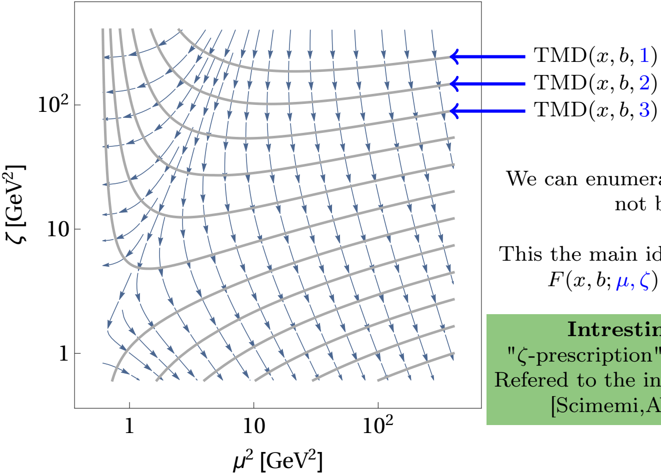
TMD distributions on the same equipotential line are equivalent.



We can enumerate them by a lines not by (μ, ζ)

This the main idea of ζ-prescription
$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$

TMD distributions on the same equipotential line are equivalent.



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This the main idea of ζ-prescription
$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$

Intresting to know:
"ζ-prescription" is an idiotic term.
Refered to the initial "naive" version [Scimemi,AV,1706.01473].

In ζ-prescription we set
$$\zeta \to \zeta_\mu(\boldsymbol{\nu}_0)$$

- TMDs are "enumerated" by $\boldsymbol{\nu}_0$ (the number of line)
- TMDs are scale independent

$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0.$$

In ζ-prescription we set
$\zeta \to \zeta_\mu(\boldsymbol{\nu}_0)$

- TMDs are "enumerated" by $\boldsymbol{\nu}_0$ (the number of line)
- TMDs are scale independent

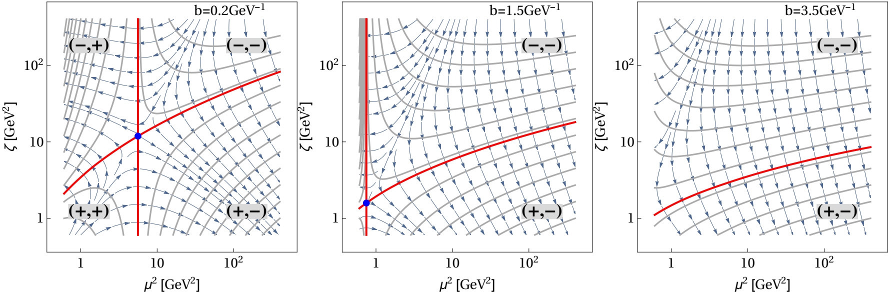
$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0.$$

- However, there is a redundant dependence on μ in the form of restriction

$$\text{at small b:} \quad F(x, b) = C(x, b, \mu_{\text{OPE}}) \otimes f(x, \mu_{\text{OPE}})$$

μ_{OPE} is restricted to the range of μ of equipotential line.

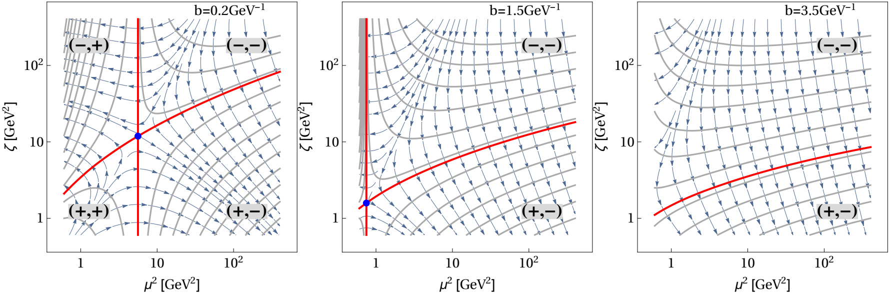
Singularities of evolution field



- Some non-interesting singularities at $\mu, \zeta \to \infty$
- Landau pole at $\mu = \Lambda$
- Saddle point (blue dot)

$$\mathcal{D}(\mu_{\text{saddle}}, b) = 0, \qquad \gamma_M(\mu_{\text{saddle}}, \zeta_{\text{saddle}}, b) = 0$$

Singularities of evolution field



- Due to presence of saddle point the set of uquipotential lines is split into subsets with restricted domains
- **Subset 1:** $\mu > \mu_{\text{saddle}}$
- **Subset 2:** $\mu < \mu_{\text{saddle}}$
- **Special line:** The one which passes though the saddle point (μ is unrestricted)
- Special lines dissect the evolution planes into quadratures of the "same evolution sign".

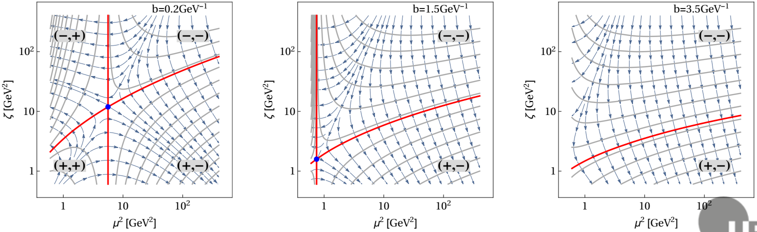
Universal scale-independent TMD

There is a unique line which passes though all μ's

The universal scale-independent TMD distribution

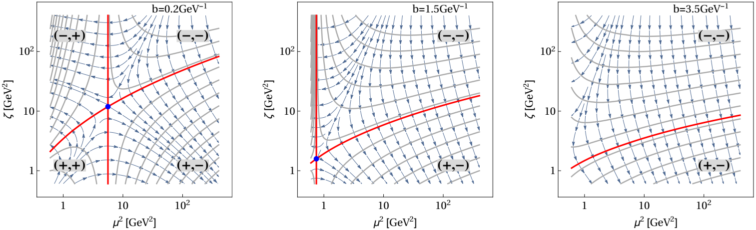
$$F(x,b) = F(x,b;\mu,\zeta_\mu)$$

where ζ_μ is the special line.



Universal scale-independent TMD

- The definition is non-perturbative
- The definition is the same for all TMD distributions (since the evolution is the same)
- **WARNING:** At large-b the saddle point can escape the observable region. Make sure that your $\mathcal{D}_{NP}$ keeps $\mu_{\text{saddle}} > \Lambda$



ζ-prescription in PT

$$\text{TMD}(x, b; \mu_i, \zeta_i) = C(x, \mathbf{L}_\mu, \mathbf{L}_{\sqrt{\zeta}}, \mu_{\text{OPE}}) \otimes \text{PDF}(x, \mu_{\text{OPE}})$$

Practically, μ_i and μ_{OPE} are both set to single μ.
1-loop example

$$C(\mu, \zeta) = \delta(\bar{x}) + a_s C_F \Bigg[-2 \underbrace{\mathbf{L}_\mu p(x)}_{\substack{\text{never large} \\ \text{thanks to} \\ \text{charge} \\ \text{conservation}}} + 2\bar{x} + \delta(\bar{x}) \Big(\overbrace{-\mathbf{L}_\mu \mathbf{L}_{\sqrt{\zeta}} + 3\mathbf{L}_\mu}^{\text{usually large}} - \zeta_2 \Big) \Bigg]$$

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ζ-prescription in PT

$$\text{TMD}(x,b;\mu_i,\zeta_i)=C(x,\mathbf{L}_\mu,\mathbf{L}_{\sqrt{\zeta}},\mu_{\text{OPE}})\otimes\text{PDF}(x,\mu_{\text{OPE}})$$

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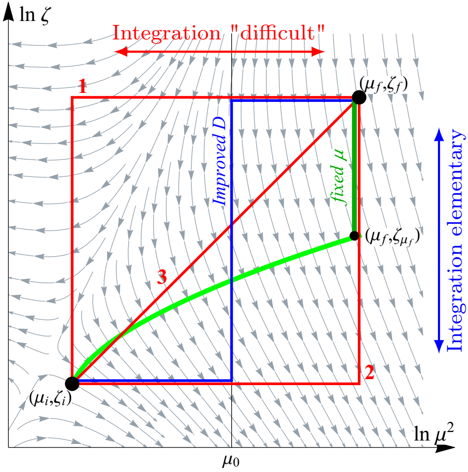
We set $\zeta\to\zeta_\mu$:

$$\zeta_\mu=\frac{2\mu}{b}e^{-\gamma_E}\overbrace{e^{3/2+a_s\ldots}}^{\text{PT-calculable}}$$

It has been used in [Scimemi,AV,1706.01473]



$$R[b;(\mu_f,\zeta_f)\to\zeta_{\mu_f}]=\left(\frac{\zeta_f}{\zeta_{\mu_f}}\right)^{-\mathcal{D}(\mu_f,b)}$$

- Numerically simple (and fast)
- $\mu_f = Q$ thus a_s is small
- Alternative form of Sudakov exponent

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TMD cross-section

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f)\{\tilde{R}^f[b;(\mu_f,\zeta_f)]\}^2 \tilde{F}_{f\leftarrow h}(x_1,b)\tilde{F}_{f'\leftarrow h}(x_2,b),$$

with

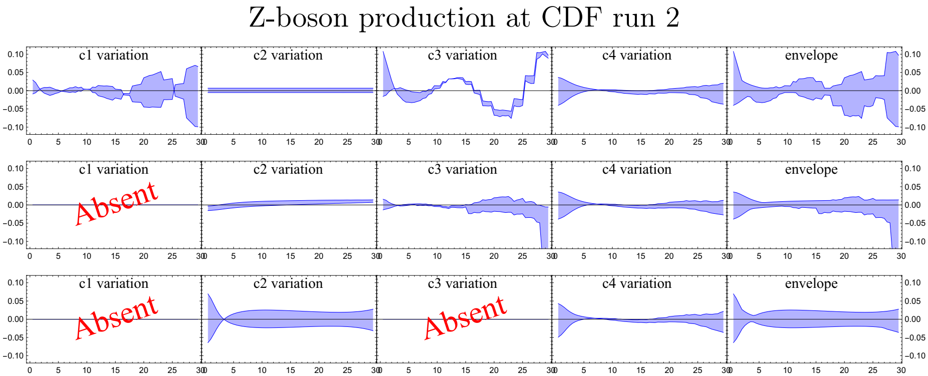
$$\tilde{R}^f[b;(\mu_f,\zeta_f)] \quad = \quad \exp\Big\{-\mathcal{D}^f_{\text{NP}}(\mu_f,b)\Big[\ln\Big(\frac{\zeta_f b}{C_0\mu_f}\Big)+v^f(\mu,b)\Big]\Big\},$$

- v is given perturbative series, $v=\frac{3}{2}+a_s\ldots$
- $\tilde{F}$ is TMD in the "naive" ζ-prescription
- There are no approximations (*ala* high energy expansion of integrals)
- The TMD at "final" point has curious form

$$F_{f\leftarrow h}(x,b;Q,Q^2) = (C_0Qb)^{-\mathcal{D}^f_{\text{NP}}(Q,b)}\, e^{-\mathcal{D}^f_{\text{NP}}(Q,b)v(Q)}\tilde{F}_{f\leftarrow h}(x,b).$$

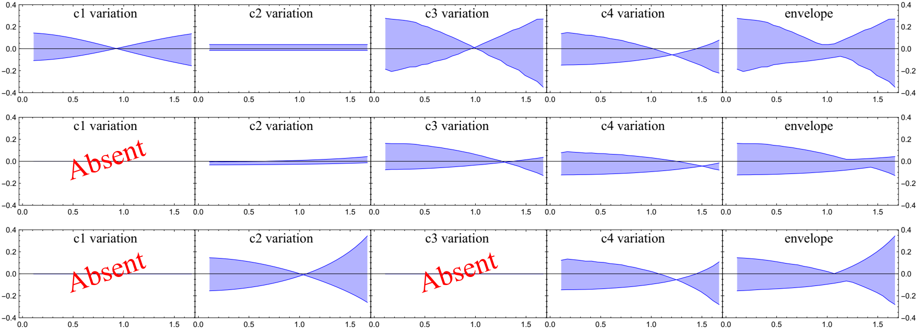
There are only 2 scales and no solution dependence

Uncertainties of TMD cross-section (1)



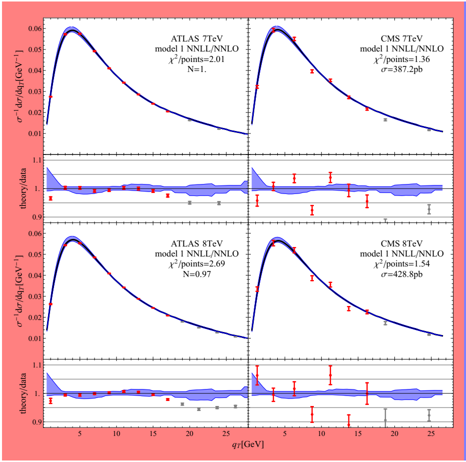
Uncertainties of TMD cross-section (2)

$$E288\ (200)\ Q = 6 - 7\ \text{GeV}$$

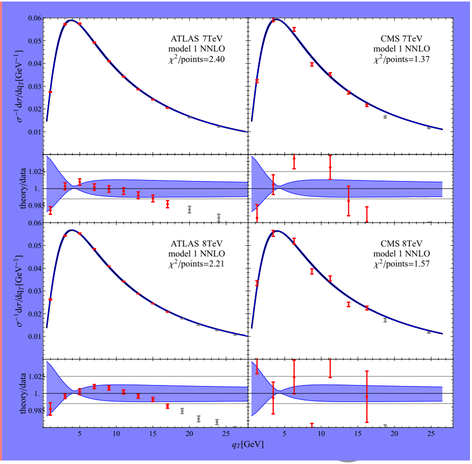


Uncertainties of TMD cross-section (3)

Glauber exponent



Optimal definition



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Conclusion

Main message:
TMD evolution is a double scale evolution.
Therefore, it should be considered with care, and then it grants many simplifications.

Message 1:
In truncated PT there is the solution-dependence of evolution

- It could be strong.
- There is no unique way to fix it.

Message 2:
TMD distributions on a same equipotential line are equivalent. Enumerate them with lines!

- Guarantied absence of (large) logarithms in coefficient function
- Universal for all quantum numbers
- Very simple practical formula (no integrations!)

Double-scale evolution is not unique for TMD case. It also appears in jet functions, k_T-resummation, joint resummation, DPDs, etc.

Backup

Collinear overlap

There are collinearly divergent subgraphs (then gluon is parallel to Wilson line), which result to overlap of UV and rapidity divergent sectors. It gives interdependance of anomalous dimension on "opposite" scale

$$\zeta \frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu),$$

$$\mu \frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu),$$

where Γ is the (light-like) cusp anomalous dimension.

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where Γ is the (light-like) cusp anomalous dimension.

Thus the logarithmic part of AD's could be fixed

(exact) $$\gamma_F(\mu,\zeta) = \Gamma(\mu)\ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_V(\mu)$$

(order-by-order) $$\mathcal{D}(\mu,b) = a_s(\mu)\frac{\Gamma_0}{2}\mathbf{L}_\mu + a_s^2\left(\frac{\Gamma_0\beta_0}{4}\mathbf{L}_\mu^2 + \frac{\Gamma_1}{2}\mathbf{L}_\mu + d^{(2,0)}\right) + ...$$

standard notation: $\mathbf{L}_X = \ln(C_0^{-2}b^2X^2)$, $C_0 = 2e^{-\gamma_E}$

Test of solution independence

$$(Q, Q^2) \to (\mu_b, \mu_b^2) \qquad \mu_b = \frac{C_0}{b} + 2\text{GeV}$$

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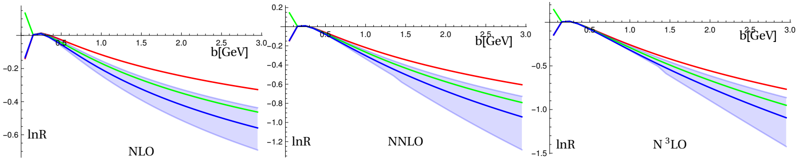
$Q = 10\text{GeV}$ (perturbation theory could work not very well)

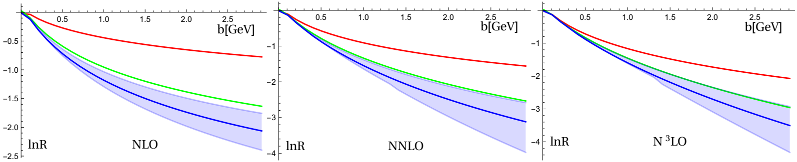


- Typical range of Fourier integration $b \in (0, 3)\text{GeV}^{-1}$
- The difference between $\ln R$ at $b = 1\text{GeV}^{-1}$ (1.74,1.39,1.23)
- The difference between R at $b = 1\text{GeV}^{-1}$ (1.09,1.08,1.06)
- Effect is almost negligible but non-zero(!)
- Improvement NLO→NNLO (~ 1.11) is (a bit) bigger then solution dependence
- Improvement NNLO→NNNLO (~ 1.04) is of the same order as solution dependence
- NP model for $\mathcal{D}$ could compensate the effect

Test of solution independence

$$(Q, Q^2) \to (\mu_b, \mu_b^2) \qquad \mu_b = \frac{C_0}{b} + 2\text{GeV}$$

$Q = M_Z$ (perturbation theory should work well)



- Typical range of Fourier integration $b \in (0,1)\text{GeV}^{-1}$
- The difference between $\ln R$ at $b = 0.5\text{GeV}^{-1}$ (2.6,1.5,1.23)
- The difference between R at $b = 0.5\text{GeV}^{-1}$ (1.6,1.35,1.18)
- Effect is very sizable, $a_s \simeq 0.009$, b in perturbative region.
- Improvement NLO→NNLO (~ 1.22) is of the same order as solution dependence
- Improvement NNLO→NNNLO (~ 1.10) is smaller then solution dependence
- NP model for $\mathcal{D}$ could not compensate the effect, it is too large in PT region.

Effects of truncation of PT

Synopsis of the problem

- There is a solution dependence of TMD evolution
- It is almost negligible at smaller Q, but large at larger Q.
- It is not disappear (or disappear very slowly) with the increase of PT order.
- At 3-loop order it is the largest uncertainty that comes from perturbation theory

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- At 3-loop order it is the largest uncertainty that comes from perturbation theory

The source of solution dependence is the violation of integrability condition.

In (truncated) perturbation theory

$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} \neq -\mu^2 \frac{d}{d\mu^2} \mathcal{D}(\mu,b) \qquad \Leftrightarrow \qquad \boldsymbol{\nabla} \times \mathbf{E} \neq 0 \tag{3}$$

The evolution flow is *non-conservative*, the scalar potential is undetermined

The TMD evolution equation has not a unique solution.

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To measure perturbative uncertainties, we typically vary scales μ.

- In exact PT, μ-dependence is absent, but at finite PT there is the perturbative mismatch between the evolution exponent and the fixed order coefficient function.
- In TMD case there is an additional source of scale-dependence, solution dependence

A TMD cross-section

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2 b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f) \times \{R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i),\mu_0]\}^2 F_{f\leftarrow h}(x_1,b;\mu_i,\zeta_i) F_{f'\leftarrow h}(x_2,b;\mu_i,\zeta_i),$$

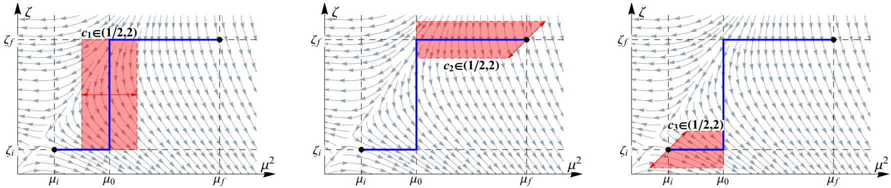
$$\mu_0 \to c_1\mu_0, \qquad \mu_f \to c_2\mu_f, \qquad \mu_i \to c_3\mu_i, \qquad \mu_{\text{OPE}} \to c_4\mu_{\text{OPE}}.$$

$$c_i \in (0.5, 2)$$

Some of these scales measure the solution dependence, some perturbative mismatch, some both.

A TMD cross-section

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2 b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f)$$
$$\times \{R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i),\mu_0]\}^2 F_{f\leftarrow h}(x_1,b;\mu_i,\zeta_i)F_{f'\leftarrow h}(x_2,b;\mu_i,\zeta_i),$$



- c_1 measure only solution dependence
- c_2 measure mismatch between H and R + solution dependence
- c_3 measure mismatch between F and R + solution dependence
- c_4 measure mismatch between C and f

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Cross-section in the improved γ

In the improved γ there is no solution dependence

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f) \{R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]\}^2 F_{f\leftarrow h}(x_1,b;\mu_i,\zeta_i) F_{f'\leftarrow h}(x_2,b;\mu_i,\zeta_i),$$

where

$$R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)] = \exp\Bigg\{ -\int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu}\Big(2\mathcal{D}^f_{\text{NP}}(\mu,b)+\gamma^f_V(\mu)\Big) + \mathcal{D}^f_{\text{NP}}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right) - \mathcal{D}^f_{\text{NP}}(\mu_i,b)\ln\left(\frac{\mu_i^2}{\zeta_i}\right)\Bigg\}.$$

There are 3 scales and no solution dependence

Cross-section in the ζ-prescription

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f)\{R^f[b;(\mu_f,\zeta_f)]\}^2 F_{f\leftarrow h}(x_1,b) F_{f'\leftarrow h}(x_2,b),$$

where

$$R^f[b;(\mu_f,\zeta_f)] \quad = \quad \exp\left\{ -\int_{\mu_{\text{saddle}}}^{\mu_f} \frac{d\mu}{\mu}\left(2\mathcal{D}^f_{\text{NP}}(\mu,b) + \gamma^f_V(\mu)\right) + \mathcal{D}^f_{\text{NP}}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right)\right\}$$

WARNING: Special line boundary condition should be taken into account in the coefficient function (details in private)
However, we can exponentiate boundary conditions and get a simple **practical formula**

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