# Generating function of web-diagrams: theory and applications

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# Introduction

**Disclaimer:** the work which I am going to present has been done in 2015, since that time I was (mainly) involved in unrelated topics.
I could forget some details...

## Plan of talk

- Generating function for web-diagrams
- Iterative substructure
- Examples of application

## Literature

- [1406.6253] (Phys.Rev.D 90 (2014)) *initial concept*
- [1501.03316] (JHEP 06 (2015) 120) *Main article*
- [1608.04920] (JHEP 12 (2016) 038) *Non-trivial example, 2loop multi-scattering SF*
- [1707.07606] (JHEP 04 (2018) 045) *Non-trivial example, 3loop decomposition + vertex reduction (see appendix)*

## Wilson lines

The method is valid for arbitrary-path Wilson lines for arbitrary gauge group

- Wilson line on arbitrary path

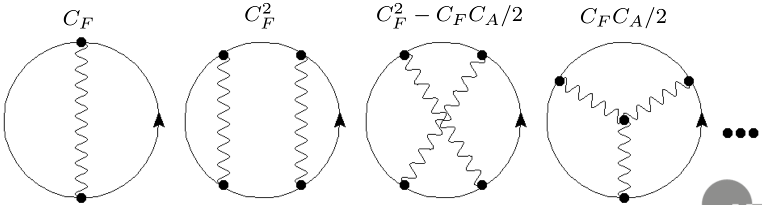
$$\mathbf{\Phi}_\gamma = P\exp\left(ig\int_0^1 d\tau\dot{\gamma}^\mu(\tau)A_\mu^A(\gamma(\tau))\mathbf{T}^A\right) \tag{1}$$

- $\mathbf{T}^A$ is the generator of gauge group. **Bold font** denotes matrices in the group space.

# Exponentiation and web-diagrams

- Original concept [Sterman,1981; Gatheral,1983; Frenkel & Taylor,1984]
- The vacuum expectation of a Wilson loop can be presented as

$$\langle \mathrm{tr}\,(\boldsymbol{\Phi}_\gamma)\rangle = \sum_{d\in\mathrm{diag.}} \underbrace{C(d)}_{\text{Color factor}}\underbrace{\mathcal{F}(d)}_{\text{Loop integral}} = \exp\left(\sum_{d\in\mathrm{diag.}} \tilde{C}(d)\mathcal{F}(d)\right)$$



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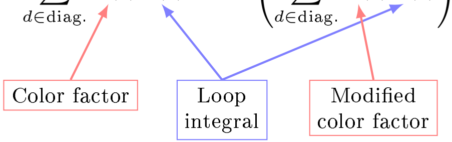
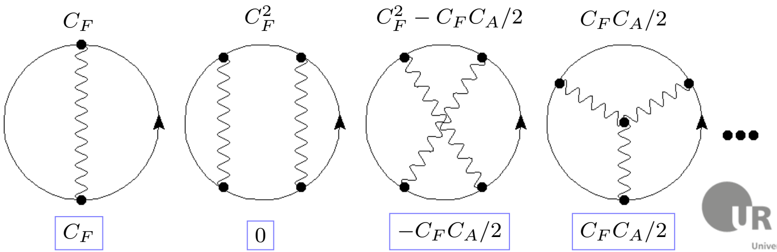




## Exponentiation and web-diagrams

$$\tilde{C}(d) = C(d) - \sum_{d'} \prod_{w \in d'} \tilde{C}(w),$$

where $w$ is **two-Wilson line irreducible graph**, also known as a **web-diagram**.

- The derivation is based on the observation that completely symmetric part of Wilson-line vertex is reducible

$$\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 A_\gamma^A(\tau_1) A_\gamma^B(\tau_2) A_\gamma^B(\tau_3) \mathbf{T}^A \mathbf{T}^B \mathbf{T}^C$$

$$\int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3 A_\gamma^A(\tau_1) A_\gamma^B(\tau_2) A_\gamma^B(\tau_3) \{\mathbf{T}^A \mathbf{T}^B \mathbf{T}^C\} + \text{anti-sym.perm.}$$

- + cyclic property of the trace, exponentiation selects maximum non-Abelian part.

### Extension for arbitrary configuration

- [Mitov,Sterman,Sung,2010] General analysis ⇒ no-simple solution
- [Gardi,Laenen,White,et al,2010 – 2014] Replica method ⇒ algorithmic method
- [AV,2015] Generating function ⟶

The core of any diagrammatic exponentiation is the expression for
**connected part of Feynman diagrams**

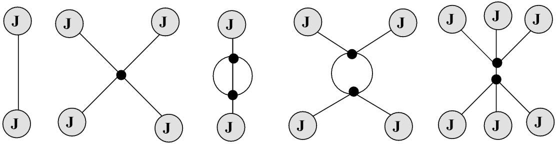
- Partition function

$$Z[J] = \int DA e^{S[A]+JO}$$

where $O$ is an operator and $J$ is a source

- The resulting expression is exponent of **only connected diagrams**

$$\frac{Z[J]}{Z[0]} = e^{W[J]}$$



## Immediate consequence

- If the operator has a form of exponent

$$O[A] = \exp\left(\int dx M(x) o[A]\right)$$

its vacuum expectation value is exponent of **only connected diagrams with operators** $o$

$$\langle O[A]\rangle = Z^{-1}[0]\int DA e^{S[A]+\int Mo[A]} = e^{W[M]},$$

where $M$ is "classical source" for operators $o$.

- $W$ is the generating function for Feynman diagrams

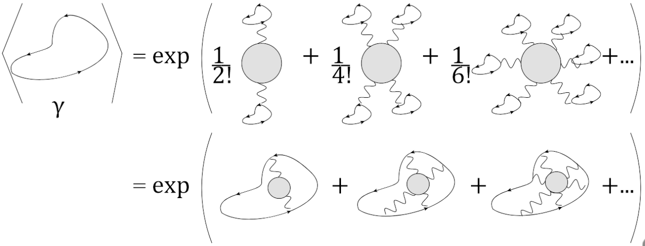
$$\begin{aligned} W[M] \quad &= \quad \int dx M(x)\langle o(x)\rangle_c + \frac{1}{2}\int dx_1 dx_2 M(x_1)M(x_2)\langle o(x_1)o(x_2)\rangle_c \\ &+\frac{1}{3!}\int dx_1 dx_2 dx_3 M(x_1)M(x_2)M(x_3)\langle o(x_1)o(x_2)o(x_3)\rangle_c + ... \end{aligned} \tag{2}$$

## Abelian exponentiation

- Abelian Wilson line is an exponent

$$\Phi_{QED} = P\exp\left(ie\int_0^1 d\tau A_\gamma(\tau)\right) = \exp\left(ie\int_0^1 d\tau A_\gamma(\tau)\right) \tag{3}$$

- Operator is $\int_0^1 d\tau A_\gamma(\tau)$
- The source is $ie$.



# Non-Abelian exponentiation

- Magnus expansion

$$\mathbf{\Phi}_\gamma = P\exp\left(ig\int d\tau A^A_\gamma(\tau)\mathbf{T}^A\right) = \exp\left(\mathbf{T}^A\sum_{n=1}^{\infty} V^A_n\right) \tag{4}$$

$$\begin{aligned}
V^A_1 &= ig\int_0^1 d\tau_0 \mathrm{tr}\left(\mathbf{T}^A\mathbf{A}_0\right)\\
V^A_2 &= -(ig)^2\int_0^1 d\tau_0\int_0^{\tau_0} d\tau_1 \mathrm{tr}\left(\mathbf{T}^A[\mathbf{A}_0,\mathbf{A}_1]\right)\\
V^A_3 &= \frac{(ig)^3}{3}\int_0^1 d\tau_0\int_0^{\tau_0} d\tau_1\int_0^{\tau_1} d\tau_2 \mathrm{tr}\left(\mathbf{T}^A\{[[\mathbf{A}_0,\mathbf{A}_1],\mathbf{A}_2]-[[\mathbf{A}_0,\mathbf{A}_2],\mathbf{A}_1]\}\right)\\
V^A_4 &= \frac{(ig)^4}{6}\int_0^1 d\tau_0\int_0^{\tau_0} d\tau_1\int_0^{\tau_1} d\tau_2\int_0^{\tau_2} d\tau_4 \times\\
&\quad \mathrm{tr}\left(\mathbf{T}^A\{[[[\mathbf{A}_1,\mathbf{A}_2],\mathbf{A}_3],\mathbf{A}_0]-[[[\mathbf{A}_0,\mathbf{A}_1],\mathbf{A}_2],\mathbf{A}_3]+[[[\mathbf{A}_0,\mathbf{A}_3],\mathbf{A}_2],\mathbf{A}_1]-[[[\mathbf{A}_2,\mathbf{A}_3],\right.\\
&\quad \ldots
\end{aligned}$$

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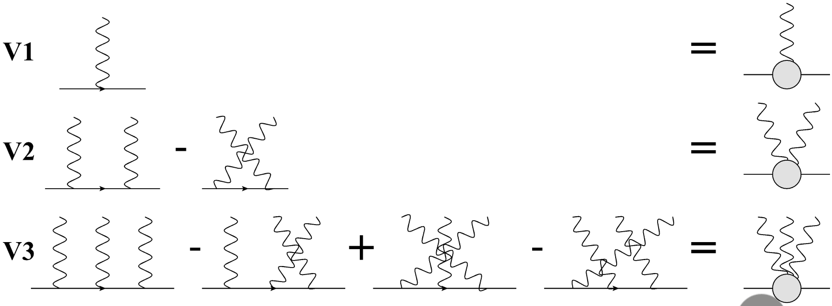


## Some properties of $V$

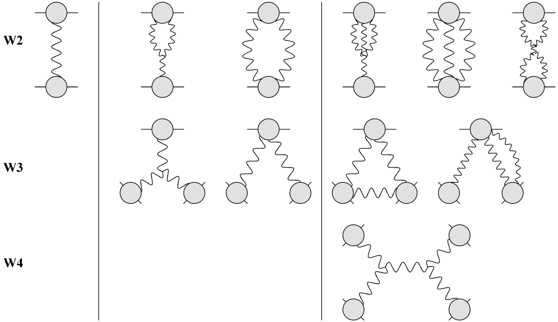
- Symmetric with respect to permutation of "legs"
- (mostly)Anti-Symmetric with respect to permutation of momenta or color
- Has lower degree of IR divergence due to cancellation of "surface divergence"

$$V_2 \rightarrow \frac{1}{k_1(k_1+k_2)} - \frac{1}{k_2(k_1+k_2)} = \frac{k_2-k_1}{k_1k_2(k_1+k_2)}$$

# Generating function for web-diagrams

$$\mathbf{W}=\sum_{n=1}^{\infty}\mathbf{W}_n,\qquad \mathbf{W}_n=\mathbf{T}^{A_1}...\mathbf{T}^{A_n}\langle V^{A_1}...V^{A_n}\rangle_c$$

- **Note 1:** **W** is completely symmetric in $A_1...A_n$
- **Note 2:** Only connected diagrams (but non-necessary Wilson-line-irreducible)
- **Note 3:** Color coefficients are maximally non-Abelian
- **Note 4:** Actually, $W$ has smaller set of diagrams then ordinary "webs"



$$\mathbf{\Phi}_\gamma = \exp\left(\mathbf{T}^A \sum_{n=1}^{\infty} V_n^A\right)$$

- $\mathbf{T}$ is source, and $V$'s are operators!

$$\langle \mathbf{\Phi}_\gamma \rangle = Z^{-1}[0] \int DAe^{iS+\mathbf{T}^A V_A} = \mathbf{Z}[\mathbf{T}]$$

There is exponentiation

$$\mathbf{Z}[T] = e^{\mathbf{W}[\mathbf{T}]}$$

$$\mathbf{\Phi}_\gamma = \exp\left(\mathbf{T}^A \sum_{n=1}^{\infty} V_n^A\right)$$

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$$\langle \mathbf{\Phi}_\gamma \rangle = Z^{-1}[0] \int DAe^{iS+\mathbf{T}^A V_A} = \mathbf{Z}[\mathbf{T}]$$

A problem:
There is no matrix exponentiation

$$\mathbf{Z}[T] \neq e^{\mathbf{W}[\mathbf{T}]}$$

Matrix

Matrix

## "Colorless" Wilson line.

- Let me screw out the matrix component of the Wilson line

$$\mathbf{\Phi}_\gamma = e^{\mathbf{T}^A V^A} = e^{\mathbf{T^A}\frac{\delta}{\delta\theta^A}} e^{\theta^B V^B}\Big|_{\theta=0}$$

Reduction exponent
Replaces $\theta$'s by $\mathbf{T}$

Colorless WL

- In this way the problem of color algebra is separated from the problem of loop-diagram computation

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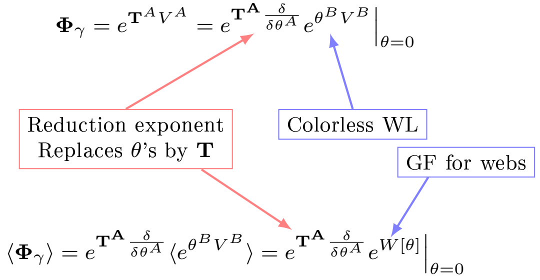

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Reduction exponent
Replaces $\theta$'s by $\mathbf{T}$

Colorless WL

GF for webs

$$\langle\mathbf{\Phi}_\gamma\rangle = e^{\mathbf{T^A}\frac{\delta}{\delta\theta^A}}\langle e^{\theta^B V^B}\rangle = e^{\mathbf{T^A}\frac{\delta}{\delta\theta^A}} e^{W[\theta]}\Big|_{\theta=0}$$

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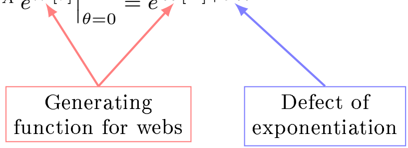
## Final form

$$\langle \boldsymbol{\Phi}_\gamma \rangle = e^{\mathbf{T^A}\frac{\delta}{\delta\theta^A}} e^{W[\theta]}\Big|_{\theta=0} = e^{\mathbf{W}[\mathbf{T}]+\delta\mathbf{W}}$$



Generating function for webs

Defect of exponentiation

- The defect is the penalty terms for matrix exponentiation
- **The defect is algebraic function of** $W$ (all order expression (4.30) in [1501.0331])
- Note : $W$ and $\delta W$ are independently gauge-invariant

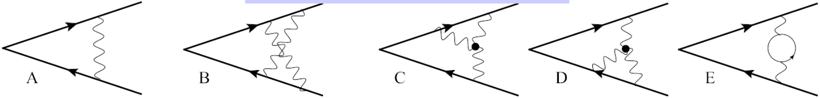
$$\delta\mathbf{W} = \sum_{n=2}^{\infty} \delta_{\mathbf{n}}\mathbf{W}, \qquad \delta_{\mathbf{n}}\mathbf{W} \sim \mathbf{W}^n$$

$$\delta_{\mathbf{2}}\mathbf{W} = \frac{\{\mathbf{W^2}\} - \mathbf{W}^2}{2}, \qquad \delta_{\mathbf{3}}\mathbf{W} = \frac{\{\mathbf{W}^3\}}{6} - \frac{\{\mathbf{W^2}\}\mathbf{W} + \mathbf{W}\{\mathbf{W^2}\}}{4} + \frac{\mathbf{W^3}}{3}$$

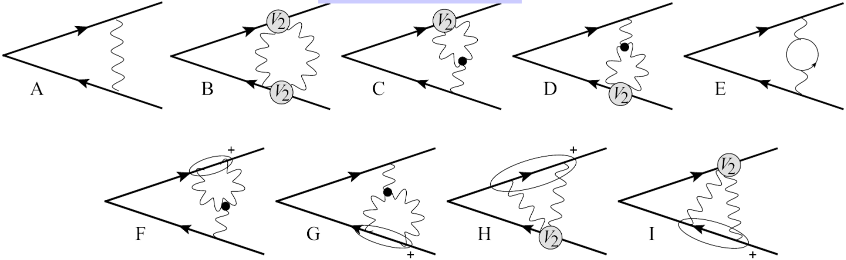
# Some applications

# Cusp of Wilson lines

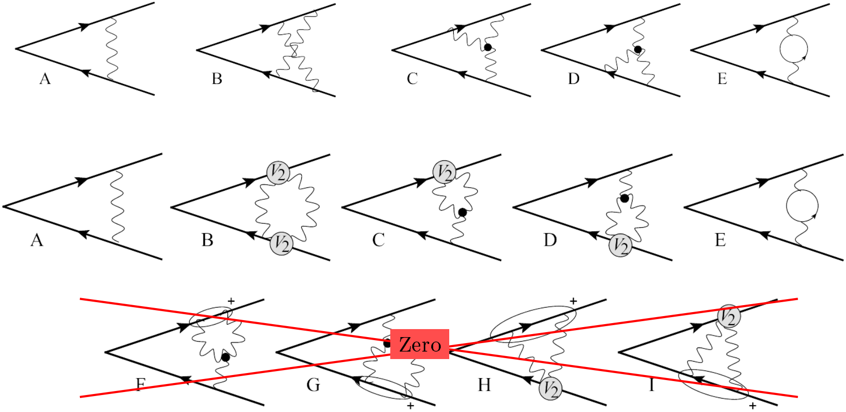
Gatheral-Frenkel web diagrams



True web diagrams



# Cusp of Wilson lines



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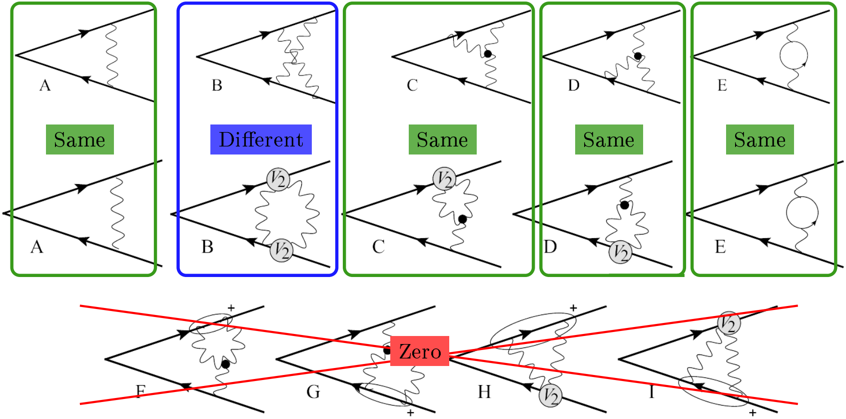

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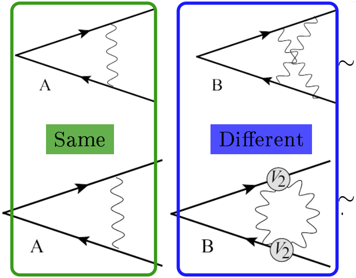
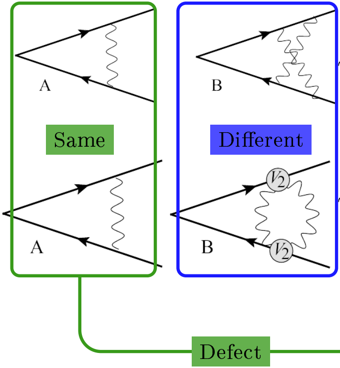


$$\sim \int_0^\infty dx_{1,2}dy_{1,2}\Delta(x_1,y_1)\Delta(x_2,y_2)\theta(x_2 > x_1)\theta(y_1 > y_2)$$

$$\sim \frac{1}{2}\int_0^\infty dx_{1,2}dy_{1,2}\Delta(x_1,y_1)\Delta(x_2,y_2)$$
$$(\theta(x_1 > x_1) - \theta(x_2 > x_1))(\theta(y_1 > y_2) - \theta(y_2 > y_1))$$

# Cusp of Wilson lines



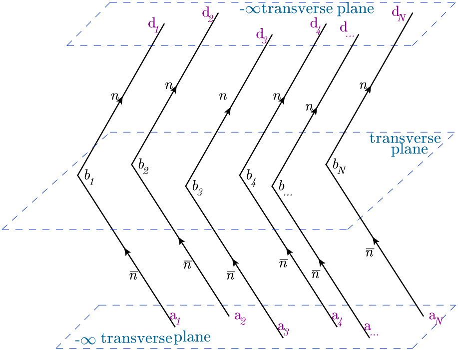
$$\sim \int_0^\infty dx_{1,2}dy_{1,2}\Delta(x_1,y_1)\Delta(x_2,y_2)\theta(x_2>x_1)\theta(y_1>y_2)$$

$$\sim \frac{1}{2}\int_0^\infty dx_{1,2}dy_{1,2}\Delta(x_1,y_1)\Delta(x_2,y_2)(\theta(x_1>x_1)-\theta(x_2>x_1))(\theta(y_1>y_2)-\theta(y_2>y_1))$$

The sum gives the known result

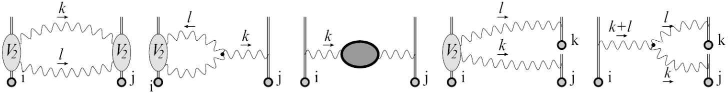
$$\delta_2 W \sim \frac{1}{2}\left(\int_0^\infty dx_1 dy_1 \Delta(x_1,y_1)\right)^2$$

# Multi-parton scattering soft-factor



- $N$ cusps of light-like Wilson lines ($n$ and $\bar{n}$ directions)
- $\mathbf{S}(b_1, ..., b_N) = \langle \mathbf{\Phi}_{cusp}(b_1)...\mathbf{\Phi}_{cusp}(b_n)\rangle$
- Equivalent to multi-jet production configuration [AV,1707.07606]
- Rapidity anomalous dimension ↔ soft anomalous dimension

$$\boldsymbol{\gamma}_S(v_1, ..., v_n) = 2\mathbf{D}(b_1, ..., b_n; \epsilon^*)$$



$$W[\theta]=\frac{a_s}{2}\sum_{j_1,j_2}\theta^A_{j_1}\theta^A_{j_2}W_{j_1,j_2}+\frac{a_s^2}{3!}\sum_{j_1j_2j_3}if^{ABC}\theta^A_{j_1}\theta^B_{j_2}\theta^C_{j_3}W_{j_1j_2j_3}+\mathcal{O}(a_s^3),$$

$$\begin{aligned}W_{ij}&=v_{ij}W[\boldsymbol{b}_{ij}],\\ W_{ijk}&=v_{ij}v_{ik}W[\boldsymbol{b}_{ij},\boldsymbol{b}_{ik},\boldsymbol{b}_{jk}].\end{aligned}$$

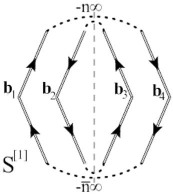
The explicit expressions for $W$ are given in [1608.04920]

### $N=2$: TMD soft factor

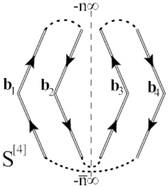
In this case the application of reduction exponent gives

$$\begin{aligned}\ln S^{\text{TMD}}&=\sigma(\boldsymbol{b})=2C_Fa_s\left(W(\boldsymbol{b})-W(\boldsymbol{0})\right)+C_FC_Aa_s^2\left(W(\boldsymbol{b},\boldsymbol{0},\boldsymbol{b})-W(\boldsymbol{0},\boldsymbol{b},\boldsymbol{b})\right)\\ &\qquad-\frac{C_FC_A}{4}a_s^2\left(3W^2(\boldsymbol{b})-4W(\boldsymbol{b})W(\boldsymbol{0})+W^2(\boldsymbol{0})\right)+\mathcal{O}(a_s^3).\end{aligned}$$

## N=4: double-scattering soft-factor



$$\ln S^{[1]} = \sigma(\boldsymbol{b}_{14}) + \sigma(\boldsymbol{b}_{23}) + \frac{1}{2}\left(\frac{C_A}{2C_F} - 1\right)(\sigma(\boldsymbol{b}_{12}) - \sigma(\boldsymbol{b}_{13}) - \sigma(\boldsymbol{b}_{24}) + \sigma(\boldsymbol{b}_{34}))^2 + \mathcal{O}(a_s^3)($$

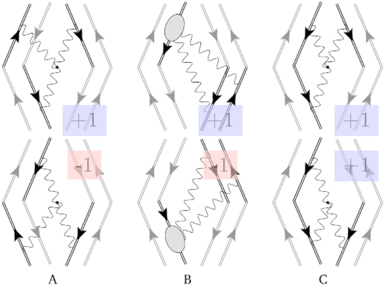


$$\begin{aligned}\ln S^{[4]} &= \sigma(\boldsymbol{b}_{12}) - \sigma(\boldsymbol{b}_{13}) + \sigma(\boldsymbol{b}_{14}) + \sigma(\boldsymbol{b}_{23}) - \sigma(\boldsymbol{b}_{24}) + \sigma(\boldsymbol{b}_{34}) \qquad (3.27)\\ &+\frac{C_A}{4C_F}(\sigma(\boldsymbol{b}_{13}) - \sigma(\boldsymbol{b}_{14}) - \sigma(\boldsymbol{b}_{23}) + \sigma(\boldsymbol{b}_{24}))(\sigma(\boldsymbol{b}_{12}) - \sigma(\boldsymbol{b}_{13}) - \sigma(\boldsymbol{b}_{24}) + \sigma(\boldsymbol{b}_{34})) + \mathcal{O}(a_s^3).\end{aligned}$$

**There is no tri-pole contribution!**

$$\mathbf{S} = \exp\left(\mathbf{T}_i^A \mathbf{T}_j^A \sigma(b_{ij}) + \text{quadrapole}\right)$$

# Absence of color odd-structures



**General proof:**
effective vertex: $V = -V(n \leftrightarrow \bar{n})$
rotation: $W = W(n \leftrightarrow \bar{n})$
$\Rightarrow W_{n\in\text{odd}} = 0$

The defect is powers of $W$
so it cannot produce odd-structures

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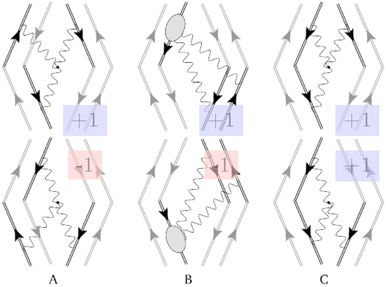

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$$\mathbf{S} = \exp\left(\sum_{n=2,4,...}^{\infty} \mathbf{T}^{A_1}...\mathbf{T}^{A_n}\sigma_{A_1...A_n}(b_{1...n})\right)$$

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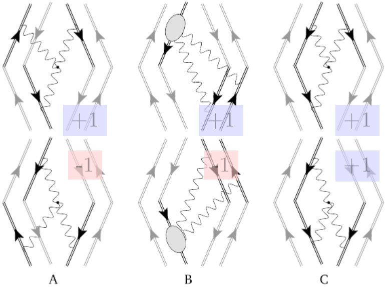

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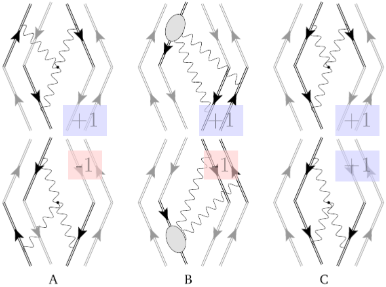
$$\mathbf{S} = \exp\Big(\sum_{n=2,4,\dots}^{\infty} \mathbf{T}^{A_1}\dots\mathbf{T}^{A_n}\sigma_{A_1\dots A_n}(b_{1\dots n})\Big)$$

$$\mathbf{D} = \sum_{n=2,4,\dots}^{\infty} \mathbf{T}^{A_1}\dots\mathbf{T}^{A_n}\mathcal{D}_{A_1\dots A_n}(b_{1\dots n})$$

# Absence of color odd-structures



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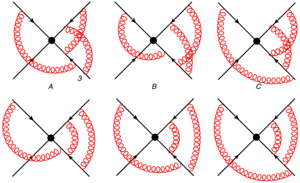
$$\mathbf{D} = \sum_{n=2,4,...}^{\infty} \mathbf{T}^{A_1}...\mathbf{T}^{A_n}\mathcal{D}_{A_1...A_n}(b_{1...n})$$

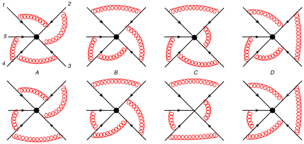
$$\boldsymbol{\gamma}_S = \sum_{n=2,4,...}^{\infty} \mathbf{T}^{A_1}...\mathbf{T}^{A_n}\gamma_{A_1...A_n}(b_{1...n})$$

[AV,1707.07606]

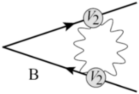
**Multiple-gluon exchange webs**(MGEWs)

MGEW = diagram with gluons coupled ONLY to WLs [Falcioni, et al, 1407.3477]





**MGEW**s are amazingly simple if WLs are on light-cone



$$=0$$

$$\sim \frac{1}{2}\int_0^\infty dx_{1,2}dy_{1,2}\Delta(x_1,y_1)\Delta(x_2,y_2)(\theta(x_1>x_2)-\theta(x_2>x_1))(\theta(y_1>y_2)-\theta(y_2>y_1))$$

For light-like Wilson line the propagator factorizes

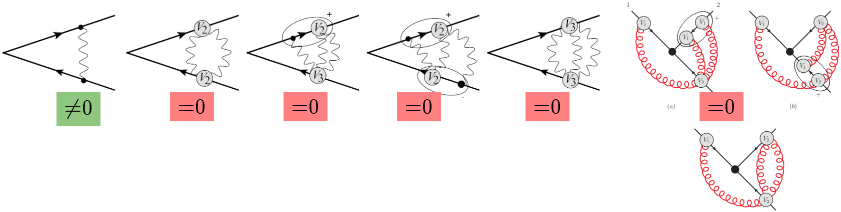
$$\Delta(x,y)\simeq\frac{(v_xv_y)}{[-(v_xx-v_yx)^2+i0]^{1-\epsilon}}=\frac{(v_xv_y)^{-\epsilon}}{2^{1-\epsilon}x^{1-\epsilon}y^{1-\epsilon}}$$

$$\Delta(x_1,y_1)\Delta(x_2,y_2)=+\Delta(x_2,y_1)\Delta(x_1,y_2)$$

but

$$(\theta(x_1>x_2)-\theta(x_2>x_1))=-(\theta(x_2>x_1)-\theta(x_1>x_2))$$

## **MGEW**s are amazingly simple if WLs are on light-cone



- The propagator structure is symmetric under permutations of any pair of coordinates
- $V_n$ has anty-symmetric part for $n > 1$
- Connected diagrams MUST contain at least 1 $V_n$ with $n > 1$
- **The generating function for MGEWs is given by a single diagram EXACTLY**

$$\text{(exact)} \qquad W^{ab}_{\text{MGEW}} = -\delta^{ab}\alpha_s(v_a \cdot v_b)^{\epsilon}\frac{\Gamma^2(\epsilon)\Gamma(1-\epsilon)}{(2\pi)^{1-\epsilon}}\left(\frac{\mu^2}{\delta^2}\right)^{\epsilon}. \tag{5}$$

## **MGEW**s to all orders

### Consequences

- In MGEW approximation expressions are generated by defect only

$$\langle \mathbf{\Phi_1 ... \Phi_n} \rangle_{\mathrm{MGEW}} = \exp\left( \underbrace{\mathbf{W}}_{\text{1-loop}} + \underbrace{\delta \mathbf{W}[\mathbf{W}]}_{\text{(n>1)-loop}} \right)$$

- MGEW approximation is gauge invariant (in a weak sense)

$$\left\langle \left(\Phi^\dagger_{v_1}\right)_{i_1 j_1} (\Phi_{v_2})_{i_2 j_2} \right\rangle\Big|_{\mathrm{MGEW}} = \exp\left[\sum_{n=1}^{\infty} \alpha_s^n v_{12}^{n\epsilon} K_1^n \left( (t^C)^a_{i_1 j_1} t^a_{i_2 j_2} E_{n,tt} + \frac{\delta_{i_1 j_1}\delta_{i_2 j_2}}{N_c} E_{n,\delta\delta} \right)\right]$$

$$\left\langle \left(\Phi^\dagger_{v_1} \Phi_{v_2}\right)_{ij} \right\rangle\Big|_{\mathrm{MGEW}} = \delta_{ij} \exp\left[ C_F \sum_{n=1}^{\infty} \alpha_s^n v_{12}^{n\epsilon} K_1^n E_{n,\mathrm{cusp}} \right]$$

| $n$ | $E_{n,tt}$ | $E_{n,\delta\delta}$ | $E_{n,\mathrm{cusp}}$ |
|---|---|---|---|
| 1 | $-1$ | 0 | $-1$ |
| 2 | $-\frac{N_c}{8}$ | 0 | $-\frac{N_c}{8}$ |
| 3 | $\frac{3-2N_c^2}{72}$ | $\frac{1-N_c^2}{48}$ | $-\frac{N_c^2}{36}$ |
| 4 | $\frac{N_c(40-33N_c^2)}{4608}$ | $\frac{5N_c(1-N_c^2)}{768}$ | $\frac{N_c(10-33N_c^2)}{4608}$ |
| 5 | $\frac{-35+93N_c^2-57N_c^4}{28800}$ | $\frac{(1-N_c^2)(23N^2-22)}{23040}$ | $\frac{N_c^2(71-114N_c^2)}{57600}$ |

# Conclusion

**Generating function for web diagrams is a powerful tool**

- Simple and handy
  - Effectively organizes the sub-sets of diagrams
- Color algebra separated from kinematics
- Allows for general analysis
  - Absence of color-odd structures in light-like ADs
  - Exact generating function for light-like MGEW

## What was not discussed

- Iterative sub-structure
- Exponentiation of cut-diagrams