Soft/rapidity anomalous dimensions correspondence & rapidity renormalization theorem

Alexey A. Vladimirov

Institut für Theoretische Physik
Universität Regensburg

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This talk is about very recent achievements.

I will talk about several seemly different processes. I will try to separate statements clearly, ask questions!

Outline of talk

- State of problem: soft factors and rapidity divergences
- Example 1: Soft factor and rapidity decomposition for TMD factorization
- Example 2: Soft factor and rapidity decomposition for Double-Drell-Yan ([AV,1608.04920])
- Soft factors for multi-parton scattering, and multi-jet production
- Soft/rapidity correspondence and its consequences ([AV,1610.05791])
- Rapidity renormalization theorem and its consequences

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General structure of the factorization theorems

The modern factorization theorems have the following general structure

$$\underbrace{\frac{d\sigma}{dX}}_{cross-X} = \underbrace{H}_{\substack{\text{Hard part} \\ \text{perturbative}}} \times \underbrace{f_1 \otimes ... \otimes J_2}_{\substack{\text{Parton distributions} \\ \text{jet-functions, etc} \\ \text{Non-pertrubative} \\ \text{universal}}} \times \underbrace{S}_{\substack{\text{Soft factor(s)} \\ \text{perturbative ?}}} + \begin{array}{c}\text{Some power} \\ \text{suppressed terms}\end{array}$$

- This is typical outcome of SCET
- For many interesting cases the individual terms in the product are singular, and requires redefinition/refactorization
- In general, the factorization task is hidden in soft factors, (they mix the singularities of different field modes)

TMD factorization

TMD factorization ($Q^2 \gg q_T^2$) gives us the following expression

$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

TMD factorization

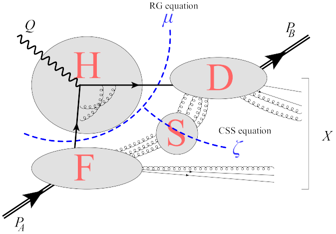
$$\frac{d\sigma}{dQdy\,d^2q_T} \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\;\; \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T) + Y$$

TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

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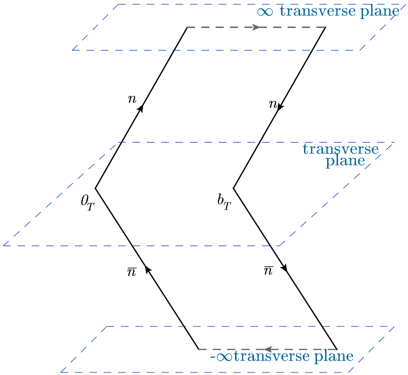
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}^\dagger_n(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n\cdot\bar{n}) = 1$$

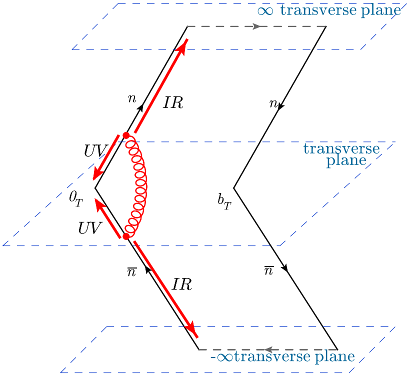
Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A^A_\mu(v\sigma + x)\mathbf{T}^A\right)$$

Multiple divergences!

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-}$$

$$= \int_0^\infty \frac{dx^+}{x^+} \int_0^\infty \frac{dy^-}{y^-}$$

$$= (\mathrm{UV}+\mathrm{IR})\,(\mathrm{UV}+\mathrm{IR})$$

Some people set it to zero.

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TMD soft factor

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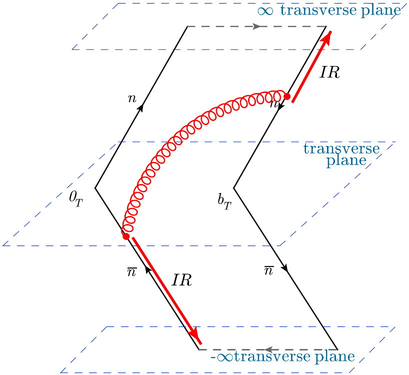

$$\begin{aligned}\int \quad & dxdy \quad D(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)} \\ = \quad & \text{IR at } x,y\to\infty\end{aligned}$$

However, it exactly cancels IR from the previous diagram
Proved at all orders,
e.g.[Echevarria,Scimemi,AV,1511.05590]

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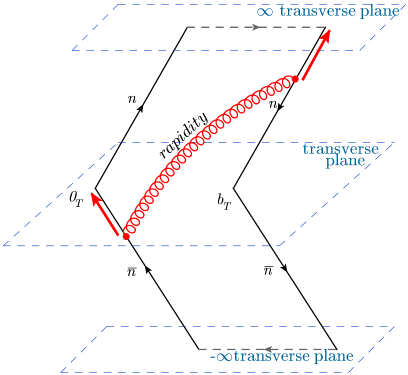
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \text{rap. div. at} \lim_{\lambda\to 0}\{x=\lambda, y=\lambda^{-1}\}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

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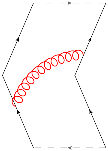
δ-regularization + dimension regularization($\epsilon > 0$)

$$P \exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)\right) \to P \exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma) e^{-\delta\sigma}\right)$$

Nice, and continent composition of regularizations, that clear separate divergences.



$$= \quad \int_0^\infty dx^+ \int_0^\infty dy^- \frac{e^{-\delta^+ y^-} e^{-\delta^- x^+}}{(2x^+y^- + \mathbf{b}_T^2)^{1-\epsilon}}$$

$x^+ \to zL, \quad y^- \to L/z$

In this calculation scheme every divergece takes particular form

$$\left(\frac{\mathbf{b}^2}{4}\right)^\epsilon \left(\ln\left(\delta^+\delta^- \frac{\mathbf{b}^2 e^{2\gamma_E}}{4}\right) - \psi(-\epsilon) - \gamma_E\right) + \left(\delta^+\delta^-\right)^{-\epsilon} \Gamma^2(-\epsilon)$$

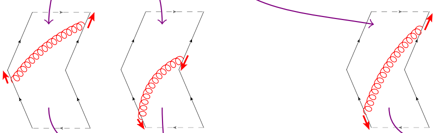
δ-regularization + dimension regularization($\epsilon > 0$)

$$P \exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)\right) \to P \exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma) e^{-\delta\sigma}\right)$$

Nice, and continent composition of regularizations, that clear separate divergences.

$x^+ \to zL, \quad y^- \to L/z$

$$\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{2LdL}{(L^2+\mathbf{b}^2)^{1-\epsilon}} e^{-L(z\delta^+ + \delta^-/z)}$$



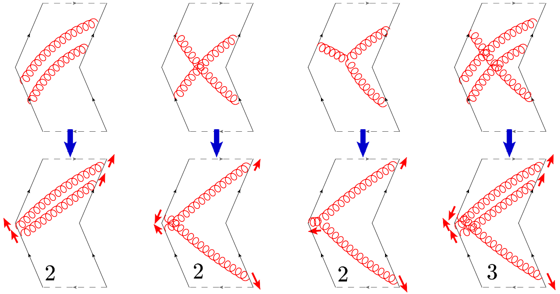
$$\left(\frac{\mathbf{b}^2}{4}\right)^\epsilon \left(\ln\left(\delta^+\delta^-\frac{\mathbf{b}^2 e^{2\gamma_E}}{4}\right) - \psi(-\epsilon) - \gamma_E\right) + \left(\delta^+\delta^-\right)^{-\epsilon}\Gamma^2(-\epsilon)$$

- Rapidity divergences happen only in one sector of diagram and independent on another
- The rules of counting the rapidity logarithms $\ln(\delta^+\delta^-)$ are very simple.

- Sub-graph contains rapidity divergence if gluon can be radiated from cusp to infinity.
- Count all divergent sub-graphs and remove them
- Repeat

Just like like UV divergences! (not accidental!)

Examples of high degree divergences

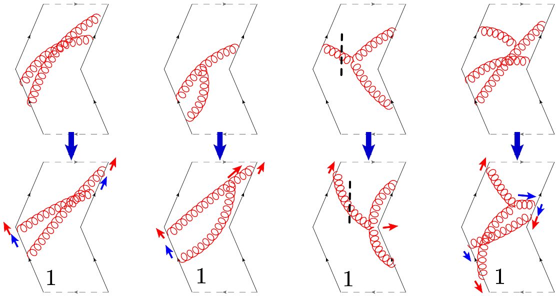


$\ln^2 \delta^+\delta^-$ $\ln^2 \delta^+\delta^-$ $\ln^2 \delta^+\delta^-$ $\ln^3 \delta^+\delta^-$

- Sub-graph contains rapidity divergence if gluon can be radiated from cusp to infinity.
- Count all divergent sub-graphs and remove them
- Repeat

Just like like UV divergences! (not accidental!)

Examples of low degree divergences



$\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$

Typical expression

Generally (say at NNLO) one expects the following form (finite ϵ, $\delta \to 0$)

$$S^{[2]} = \overbrace{A_1\boldsymbol{\delta}^{-2\epsilon} + A_2\boldsymbol{\delta}^{-\epsilon}\boldsymbol{B}^{\epsilon}}^{\text{IR}} + \underbrace{\boldsymbol{B}^{2\epsilon}\Big(A_3\ln^2(\boldsymbol{\delta}B)}_{\text{cancel in sum of diagram}} + A_4\ln(\boldsymbol{\delta}B) + A_5\Big)$$

- Terms $\sim (\boldsymbol{\delta})^{-\epsilon}$ cancel exactly at all orders (proved!)
- A_3 cancels due to Ward identity (alike leading UV pole for cusp)(what about NNNLO?)

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$(proved?)

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

It allows to split rapidity divergences and define individual TMDs.

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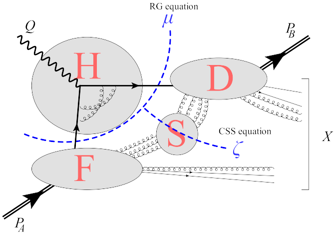
$$\exp(A\ln(\delta^+\delta^-)+B)=\exp\left(\frac{A}{2}\ln((\delta^+)^2\zeta)+\frac{B}{2}\right)\exp\left(\frac{A}{2}\ln((\delta^-)^2\zeta^{-1})+\frac{B}{2}\right)$$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\;\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities

$$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\;F(z_1,b_T;\zeta^+)\;D(z_2,b_T;\zeta^-)+Y$$



TMD PDF	TMD FF
$\sqrt{S}\Phi_{h_1}$	$\sqrt{S}\Delta_{h_2}$
(regular)	(regular)

The extra "factorization" introduces extra scale ζ.
And corresponded evolution equation

$$\zeta\frac{d}{d\zeta}F=\frac{A}{2}F=-\mathcal{D}F$$

Rapidity anomalous dimension

- The factorization of rapidity singularities in the TMD case can be shown indirectly at $b \to 0$, using collinear factorization for (integrated) DY and symmetry arguments.
- Checked at NNLO (two-loops) [Echevarria,Scimemi,AV,1511.05590],[Lübbert,Oredsson,Stahlhofen,1602.01829], [Li,Neill,Zhu,1604.00392]
- The direct all-order proof is absent (but will be presented here.).
- Nothing is known about large (finite) b (however, leading renormalon part also factorizes [Scimemi,AV,1609.06047]).

Let me present another, less practical, but more general example.

Double-Drell-Yan scattering

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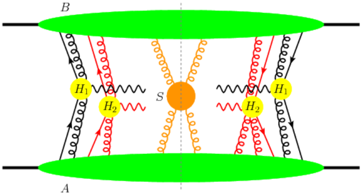


pictures from [1510.08696]

Double Drell-Yan scattering

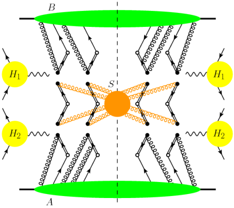
- Experimental status is doubtful
- Factorization is proved (in the weak form) [Diehl,et al,1510.08696]
- In many aspects similar to TMD factorization (SCET II)
- The same problem of rapidity factorization

DPD soft factor (very singular)

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\; F_{h1}^{A}(z_{1,2}, b_{1,2,3,4}) S^{AB}(b_{1,2,3,4}) \bar{F}_{h2}^{B}(z_{1,2}, b_{1,2,3,4}) + Y$$

power suppressed terms

DPD (singular)



Structure is similar to TMD Drell-Yan but now it contains COLOR

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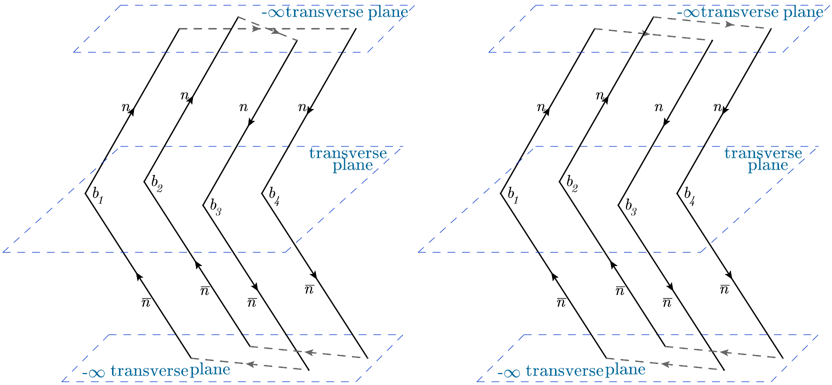
Color structure makes a lot of difference

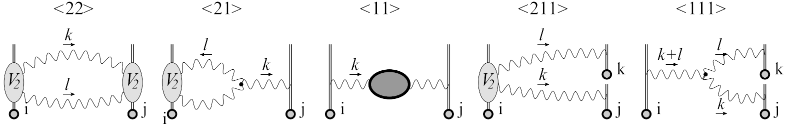
$$F^A_{h1} S^{AB} \bar{F}^B_{h2} \xrightarrow{\text{singlets}} (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix}$$

- Soft-factors $S^{\mathbf{ij}}$ are sum of Wilson loops and double Wilson loops (all possible connections).
- Soft-factors non-zero even in the integrated case.



Evaluation at NNLO [AV,1608.04920]

- Brute force evaluation would lead a lot of (similar) diagrams.
- The better way is to compute the generating function [AV, 1406.6253, 1501.03316]



- All non-trivial three-Wilson line interactions cancel!
- The final result expresses exactly via TMD soft factor only!

$$\text{TMD SF} \quad : \quad \ln S^{\rm TMD} = \sigma(\mathbf{b})$$

$$\text{Single loop SF} \quad : \quad \ln S^{[4]} = \sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})+\sigma(\mathbf{b}_{14})+\sigma(\mathbf{b}_{23})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}) + \frac{C_A}{4C_F}(\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{14})-\sigma(\mathbf{b}_{23})+\sigma(\mathbf{b}_{24}))(\sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}))$$

$$\text{Double loop SF} \quad : \quad \ln S^{[1]} = \sigma(\mathbf{b}_{14})+\sigma(\mathbf{b}_{23})+\frac{1}{2}\Big(\frac{C_A}{4C_F}-1\Big)(\sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}))^2$$

- This structure is independent on regularization procedure!

REMINDER: TMD factorization

$$S^{\mathrm{TMD}} = e^{\sigma(\mathbf{b})} = e^{\sigma^+(\mathbf{b})}e^{\sigma^-(\mathbf{b})}, \qquad \sigma^{\pm} = \frac{A}{2}\ln((\delta^+)^2\zeta^{\pm 1}) + \frac{B}{2}$$

Matrix factorization of rapidity divergences

Using the decomposition above, inserting it into DPD SF we obtain matrix relation

$$S^{\mathrm{DPD}} = s^T(\ln(\delta^+)) \cdot s(\ln(\delta^-))$$

$$s = \exp\left[\begin{pmatrix} A^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ A^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix}\ln(\delta) + \begin{pmatrix} B^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ B^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix}\right]$$

$A^{\mathbf{ij}}$ and $B^{\mathbf{ij}}$ are rather complicated non-linear compositions of TMD's A and B

Finalizing DPD factorization

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\ (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix} + Y$$

spliting rapidity singularities
$S(b_{1,2,3,4}) \to s^T(b_{1,2,3,4};\zeta^+) s(b_{1,2,3,4};\zeta^-)$

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}]\, e^{-i(qb)_T} H_1(Q_1^2) H_2(Q_2^2)\ F(z_{1,2}, b_{1,2,3,4};\zeta^+) \bar{F}(z_{1,2}, b_{1,2,3,4};\zeta^-) + Y$$

DPD $(sF_{h_1})^T$ (regular)

DPD sF_{h_2} (regular)

Matrix rapidity evolution

$$\frac{dF(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)}{d\ln\zeta} = -F(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)\mathbf{D}(\mathbf{b}_{1,2,3,4},\mu)$$

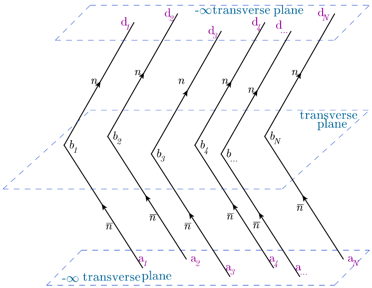
where $\mathbf{D}$ is matrix build (linearly) of TMD rapidity anomalous dimensions. E.g.

$$D^{\mathbf{18}} = \frac{\mathcal{D}(\mathbf{b}_{12}) - \mathcal{D}(\mathbf{b}_{13}) - \mathcal{D}(\mathbf{b}_{24}) + \mathcal{D}(\mathbf{b}_{34})}{\sqrt{N_c^2 - 1}}$$

Let's look at multi-parton scattering

- Just as double-parton, but multi..(four WL's → arbitrary number WL's)
- Factorization can be/is proven [Diehl, et al,1510.08696,1111.0910]
- Too many color-singlets, better to work with explicit color indices (color-multi-matrix)

$$\Sigma^{(a_1...a_N);(d_1...d_N)}(\mathbf{b}_1,...,\mathbf{b}_N) = \mathbf{\Sigma}(\mathbf{b}_{1,..,N})$$



Result at NNLO is amazingly simple

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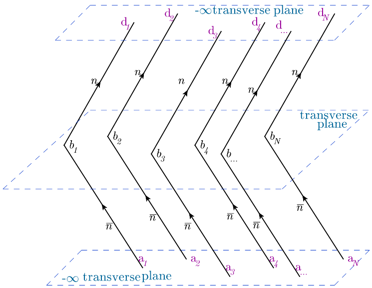

Result at NNLO is amazingly simple

$$\mathbf{\Sigma}(\mathbf{b}_{1,..,N}) = \exp\left(-\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A\sigma(\mathbf{b}_{ij}) + \mathcal{O}(a_s^3)\right)$$

- $\mathbf{T}_i^A\mathbf{T}_j^A$ = "dipole"
- $\mathcal{O}(a_s^3)$ contains also "color-multipole" terms
- Rapidity factorization for dipole part is straightforward (assuming TMD factorization)

Main question

- Can rapidity factorization be proven at all orders?

Main lessons

- Rapidity divergences characterize interaction of origin with infinity. Do not depend on the rest!
- Has very similar (topologically) structure with UV divergences.

Hint

- The color dependent expression reminds us something.... Multi-particle production!

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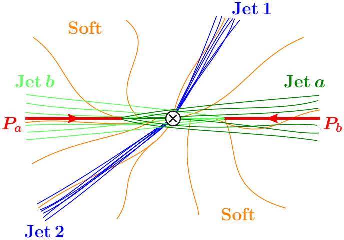
END OF INTRODUCTION

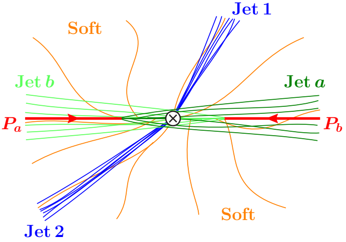
Multi-jet production



picture from
[Stewart,Tackmann,Waalewijn,0910.0467]

$$\frac{d\sigma}{dX} = H^{IJ} f_A \otimes f_B \otimes J_1 \otimes J_2\, S^{JI}$$

Multi-jet production



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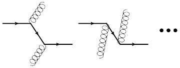
Hard part: pertrubative

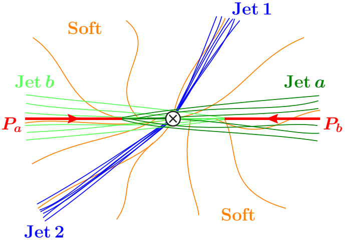


Parton distributions
$f \sim \langle h|\bar{q}[\text{Wilson lines}]q|h\rangle$

$$\frac{d\sigma}{dX} = H^{IJ}\, \overbrace{f_A \otimes f_B} \otimes \underbrace{J_1 \otimes J_2}\, \mathcal{S}^{JI}$$

Jet functions
$J \sim \langle 0|q|X_J\rangle\langle X_J|\bar{q}|0\rangle$

Multi-jet production



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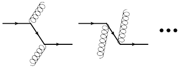
Hard part: pertrubative



Parton distributions
$f \sim \langle h|\bar{q}[\text{Wilson lines}]q|h\rangle$

$$\frac{d\sigma}{dX} = H^{IJ}\, f_A \otimes f_B \otimes J_1 \otimes J_2\, \mathcal{S}^{JI}$$

Jet functions
$J \sim \langle 0|q|X_J\rangle\langle X_J|\bar{q}|0\rangle$

Soft factor

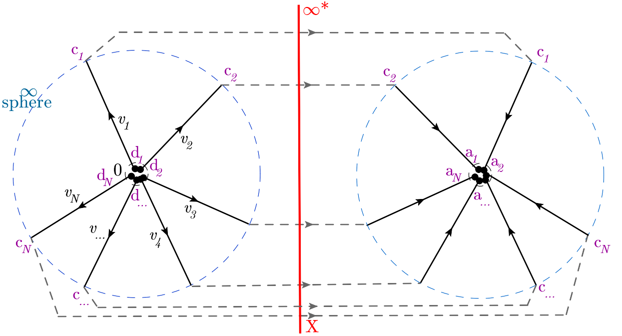
- The details of definition for soft factors differs from process to process. Generally: $S \sim \langle 0|[[\text{Wilson lines}]|0\rangle$
- The soft factor can be product of soft factors
- Soft factors also color matrix (but coupled to hard part).

Structure of N-jet soft factor

$$\mathcal{S}^{\{ad\}}(\{v\}) = \sum_X w_X \Pi_X^{\dagger\{ac\}}(\{v\})\Pi_X^{\{cd\}}(\{v\})$$ Soft factor

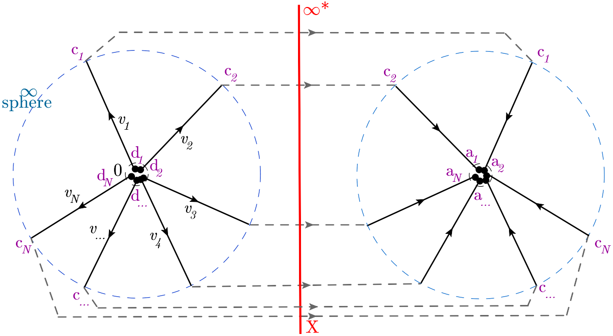
$$\Pi_X^{\{cd\}}(\{v\}) = \langle X|T[\Phi_{v_1}^{c_1 d_1}(0)...\Phi_{v_N}^{c_N d_N}(0)]|0\rangle$$

Wilson "ray" $$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$



Structure of generic soft factor

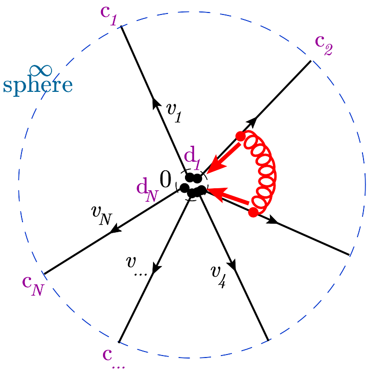
- Depending on w_X soft factors can have very different structure
- Such structure of soft factor appears in many places: multi-jet product, event shapes, hard-collinear factorization and Sudakov factorization, threshold resummation.
- In the most popular configurations the soft factor (or its parts) have been evaluated up to N^{3}LO
- In the following I consider only $v_i^2 = 0$ (massless partons)



Soft anomalous dimension

- With the proper choice of w_X the multi-jet soft factor is IR finite.
- UV divergences are renormalized by the appropriate matrix

$$\mathbf{S}^{\text{ren}}(\{v\}) = \mathbf{Z}^\dagger(\{v\})\mathbf{S}(\{v\})\mathbf{Z}(\{v\})$$



Renormalization factor knows only about $\mathbf{\Pi}$

$$\mathbf{S}^{\text{ren}}(\{v\}) = \sum_X w_X \mathbf{\Pi}_X^{\dagger\text{ren}}(\{v\})\mathbf{\Pi}_X^{\text{ren}}(\{v\})$$

$$\mathbf{\Pi}_X^{\text{ren}}(\{v\}) = \mathbf{\Pi}_X^{\text{ren}}(\{v\})\mathbf{Z}(\{v\})$$

- Individually $\mathbf{\Pi}$ is horrible object
  - Not gauge invariant
  - IR singular
  - Dependent on X
- However UV factor $\mathbf{Z}$ is well-defined
  - Gauge invariant
  - Independent on X

- The RG anomalous dimension for operator $\mathbf{\Pi}$ is called "soft anomalous dimension"

$$\mu^2 \frac{d}{d\mu^2} \mathbf{\Pi}(\{v\}) = \mathbf{\Pi}(\{v\}) \gamma_s(\{v\})$$

The soft anomalous dimension is subject of intensive studies

- It is known up to 3-loops (4-loops in $\mathcal{N} = 4$ SYM)
- It is conformally invariant (rescaling of the vectors $\{v\}$)
- It has structure (dipole part)

$$\gamma_s(\{v\}) = -\sum_{i<j} \mathbf{T}_i^A \mathbf{T}_j^A \tilde{\gamma}(v_i \cdot v_j) + \ldots$$

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$$\gamma_s(\{v\}) = -\sum_{i<j} \mathbf{T}_i^A \mathbf{T}_j^A \tilde{\gamma}(v_i \cdot v_j) + ...$$

Compare with the rapidity anomalous dimension (dipole part)

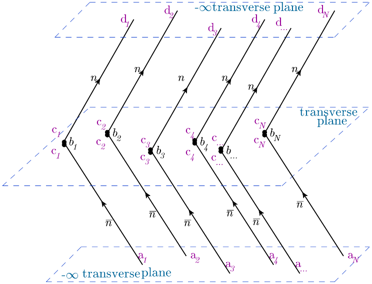
$$\mathbf{D}(\{b\}) = -\sum_{i<j} \mathbf{T}_i^A \mathbf{T}_j^A \mathcal{D}((b_i - b_j)^2) + ...$$

Rewriting multi-parton scattering soft factor

$$\Sigma^{\{ad\}}(\{\mathbf{b}\}) = \langle 0|T[\left(\Phi_{-\bar{n}}^{a_1c_1}\Phi_{-n}^{\dagger c_1d_1}\right)(\mathbf{b}_1)...\left(\Phi_{-\bar{n}}^{a_Nc_N}\Phi_{-n}^{\dagger c_Nd_N}\right)(\mathbf{b}_N)]|0\rangle$$ MPS Soft factor

$$\mathbf{\Sigma}(\{\mathbf{b}\}) = \sum_X \tilde{w}_X \mathbf{\Xi}^\dagger_{\bar{n},X}(\{\mathbf{b}\})\mathbf{\Xi}_{n,X}(\{\mathbf{b}\})$$

$$\Xi^{\{cd\}}_{n,X}(\{\mathbf{b}\}) = \langle X|T[\Phi_{-n}^{\dagger c_1d_1}(\mathbf{b}_1)...\Phi_{n}^{\dagger c_Nd_N}(\mathbf{b}_N)]|0\rangle$$

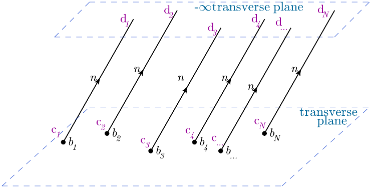


Rewriting multi-parton scattering soft factor

$$\Sigma^{\{ad\}}(\{\mathbf{b}\}) = \langle 0|T[\left(\Phi^{a_1c_1}_{-\bar{n}}\Phi^{\dagger c_1d_1}_{-n}\right)(\mathbf{b}_1)...\left(\Phi^{a_Nc_N}_{-\bar{n}}\Phi^{\dagger c_Nd_N}_{-n}\right)(\mathbf{b}_N)]|0\rangle \quad \text{MPS Soft factor}$$

$$\mathbf{\Sigma}(\{\mathbf{b}\}) = \sum_X \tilde{w}_X \mathbf{\Xi}^\dagger_{\bar{n},X}(\{\mathbf{b}\})\mathbf{\Xi}_{n,X}(\{\mathbf{b}\})$$

$$\Xi^{\{cd\}}_{n,X}(\{\mathbf{b}\}) = \langle X|T[\Phi^{\dagger c_1d_1}_{-n}(\mathbf{b}_1)...\Phi^{\dagger c_Nd_N}_{n}(\mathbf{b}_N)]|0\rangle$$



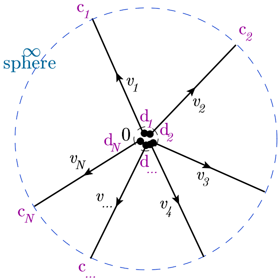
Unnatural act

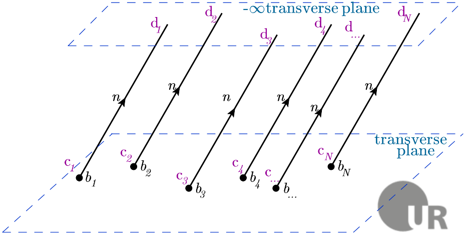
In many aspects $\mathbf{\Xi}$ is similar to $\mathbf{\Pi}$

- Individually $\mathbf{\Xi}$ is horrible object
  - Not gauge invariant
  - IR singular
  - End-point singularities
  - Dependent on X
- However it is as horrible as $\mathbf{\Pi}$

Conformal-Stereographic transformation

$$\mathcal{C}: \quad \{x^+, x^-, \mathbf{x}_T\} \longrightarrow \left\{-\frac{1}{2x^+}, x^- - \frac{\mathbf{x}_T^2}{2x^+}, \frac{\mathbf{x}_T}{\sqrt{2}x^+}\right\}$$





Conformal-Stereographic transformation

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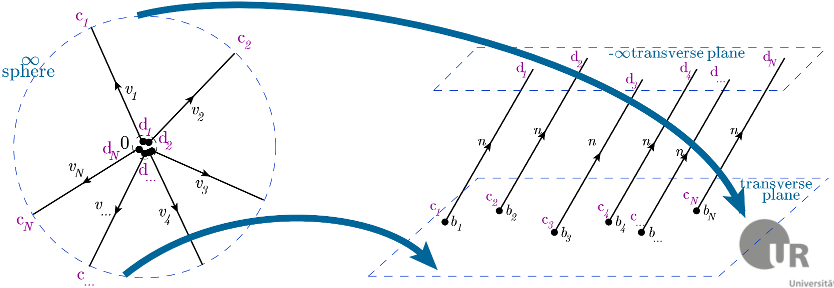
- ∞-sphere transforms to the transverse plane (at origin)



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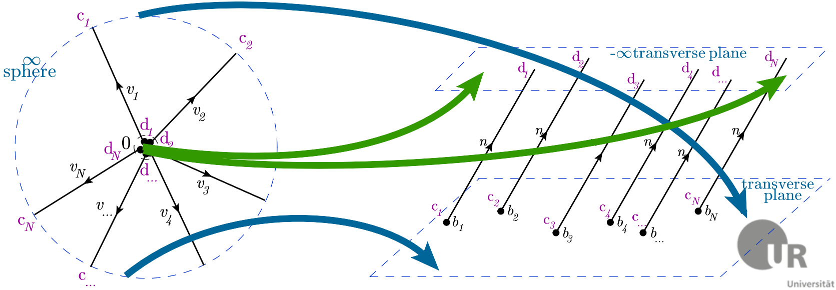
- ∞-sphere transforms to the transverse plane (at origin)
- 0 transforms to the transverse plane (at $-\infty n$)



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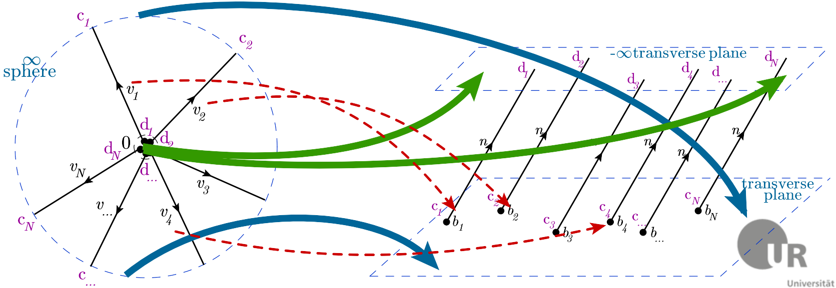
- ∞-sphere transforms to the transverse plane (at origin)
- 0 transforms to the transverse plane (at $-\infty n$)
- Scalar products $2(v_i \cdot v_j)$ transforms to $(\mathbf{b}_i - \mathbf{b}_j)^2$



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Conformal-Stereographic transformation

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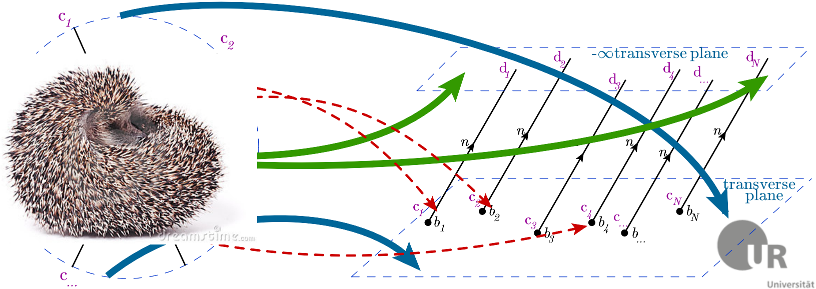

Conformal-Stereographic transformation

$$\mathcal{C}: \quad \{x^+, x^-, \mathbf{x}_T\} \longrightarrow \left\{ -\frac{1}{2x^+}, x^- - \frac{\mathbf{x}_T^2}{2x^+}, \frac{\mathbf{x}_T}{\sqrt{2}x^+} \right\}$$

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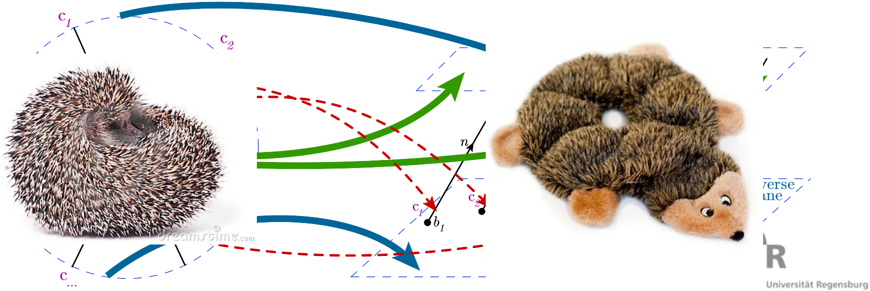
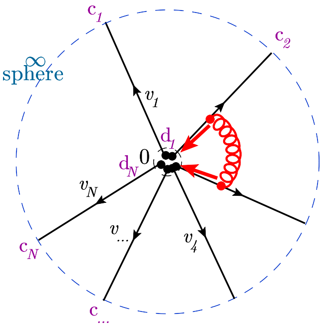


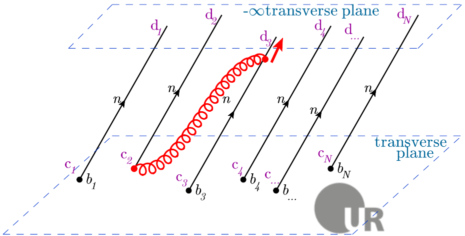
Conformal-Stereographic transformation

$$\mathcal{C}\Phi_v^{cd}(0) \;=\; \Phi_{-n}^{\dagger cd}\left(\frac{\mathbf{v}_T}{\sqrt{2}v^+}\right)$$

$$\mathbf{\Pi}_{X=0}\left(\{v\}\right) \longrightarrow \mathbf{\Xi}_{X=0}\left(\{\mathbf{b}\}\right)$$

- UV divergences of $\mathbf{\Pi}$ map onto the rapidity divergences $\mathbf{\Xi}$
- Rapidity divergent part of $\mathbf{\Xi}$ is gauge invariant and independent on X (like UV of $\mathbf{\Pi}$)





In conformal field theory

$$\mathbf{\Pi}_X(\{v\})\mathbf{Z}(\{v\}) \xrightarrow{\mathcal{C}} \mathbf{\Xi}_X(\{\mathbf{b}\})\mathbf{R}(\{\mathbf{b}\})$$

- In conformal field theory rapidity divergences renormalizable (for every factor $\mathbf{\Xi}$).
- $\mathbf{R}$ can be obtained from $\mathbf{Z}$ (by some transformation of regularizations)

$$\text{UV RGE:} \quad \mu^2\frac{d}{d\mu^2}\mathbf{\Pi}_X = \mathbf{\Pi}_X\boldsymbol{\gamma}_s \xrightarrow{\mathcal{C}} \zeta\frac{d}{d\zeta}\mathbf{\Xi}_X = 2\mathbf{\Xi}_X\mathbf{D} \quad \text{:Rap. RGE}$$

In conformal field theory

$$\mathbf{\Pi}_X(\{v\})\mathbf{Z}(\{v\}) \xrightarrow{\mathcal{C}} \mathbf{\Xi}_X(\{\mathbf{b}\})\mathbf{R}(\{\mathbf{b}\})$$

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$$\text{UV RGE:} \quad \mu^2 \frac{d}{d\mu^2}\mathbf{\Pi}_X = \mathbf{\Pi}_X \boldsymbol{\gamma}_s \xrightarrow{\mathcal{C}} \zeta\frac{d}{d\zeta}\mathbf{\Xi}_X = 2\mathbf{\Xi}_X\mathbf{D} \quad \text{:Rap. RGE}$$

Since anomalous dimensions independent on regularization we have simple relation

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\})$$

Indeed, such equality has been recently observed at NNNLO in $\mathcal{N}=4$ SYM [Li,Zhu,1604.01404]

QCD at critical coupling

Conformal symmetry of QCD is restored at critical coupling a^*

In dimensional regularization

At critical coupling β-function vanish ($a_s = g^2/(4\pi)^2$)

$$\beta(g) = g\left(-\epsilon - a_s\beta_0 - a_s^2\beta_1 - ...\right),$$

Equation $\beta(g^*) = 0$ defines the value of $a_s^*(\epsilon)$ or equivalently defines the number of dimensions in which QCD critical

$$\beta(\epsilon^*) = 0 \qquad \longrightarrow \qquad \epsilon^* = -a_s\beta_0 - a_s^2\beta_1 - ...$$

In MS-like schemes UV anomalous dimension is independent on definition of ϵ.

UV anomalous dimensions are conformal invariant

[Vasiliev; 90's]
for modern applications see e.g.
[Braun,Manashov,1306,5644]

Soft/rapidity anomalous dimension correspondence

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

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Soft/rapidity anomalous dimension correspondence

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

In QCD

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- Seems to be absolutely unique relation
- Exact relation!
- Connects different regimes of QCD
- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$

→ Lets test it.

TMD rapidity anomalous dimension

- $N = 2$, no matrix structure, $\boldsymbol{B} = \frac{b^2}{4}$, $L = \ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right) \leftrightarrow \ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)} = L + 0$

NLO | NLO

$-2\left(\boldsymbol{B}^{\epsilon}\Gamma(-\epsilon) + \frac{1}{\epsilon}\right) = \mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}$ $\quad$ $a_s\mathcal{D}^{(1)}$

$$\gamma_s(\{v\}) = 2\mathcal{D}(\{\mathbf{b}\}, \epsilon^*)$$

Expand in a_s

$$a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{\nu^2}\right) + 0\right) = a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{4e^{-2\gamma_E}}\right) + 0\right)$$

Obvious relation, QCD is confirmal at leading order.

$\nu^2 = 4e^{-2\gamma_E}$

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TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\quad \boldsymbol{B}=\frac{b^2}{4},\ L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow \ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+\ldots)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$

NLO | NLO

N^{2}LO

$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}$

$a_s\mathcal{D}^{(1)}+a_s^2?\mathcal{D}^{(2)}$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$..+a_s^2\left(\Gamma_2 L+\gamma^{(2)}\right)=..+2a_s^2\left(\mathcal{D}^{(2)}-2\beta_0(L^2+\zeta_2)\right)$$

We found 2-loop rapidity anomalous dimension

$$\mathcal{D}^{(2)}_{L=0}=\left(\frac{404}{27}-14\zeta_3\right)C_A-\frac{112}{27}\frac{N_f}{2}$$

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TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\quad \boldsymbol{B}=\frac{b^2}{4},\ L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow \ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$$\gamma_s^{(1)}=L+0$$

$$\gamma_s^{(2)}=[(\tfrac{67}{9}-2\zeta_2)C_A-\tfrac{20}{18}N_f]L+ (28\zeta_3+\dots)C_A+(\tfrac{112}{27}-\tfrac{4}{3}\zeta_2)N_f$$

$$\gamma_s^{(3)}=[\tfrac{245}{3}C_A^2+\dots]L+ +(-192\zeta_5C_A^2+\dots+\tfrac{2080}{729}N_f^2)$$

NLO | NLO

N^{2}LO | N^{2}LO

N^{3}LO

$$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\tfrac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$$

$$\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)=\mathcal{D}^{(2)} +\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\Big(\frac{3(4-3\epsilon)}{2\epsilon}C_A-N_f\Big)\Big) +\boldsymbol{B}^\epsilon\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0+\frac{\beta_0}{2\epsilon^2}-\frac{\Gamma_1}{2\epsilon}$$

[Echevarria,Scimemi,AV,1511.05590]

$$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}+a_s^3\gamma_s^{(3)} \rightarrow \gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*) \leftarrow a_s\mathcal{D}^{(1)}+a_s^2\mathcal{D}^{(2)}+a_s^3?\mathcal{D}^{(3)}$$

Expand in a_s

$$..+a_s^3\left(\Gamma_3L+\gamma^{(3)}\right)=..+2a_s^2\Big[\mathcal{D}^{(3)}-\frac{2\beta_0^2}{3}L^3-\left(\frac{\beta_0\Gamma_1}{2}+\beta_1\right)L^2+\beta_0(\gamma_1-2\beta_0\zeta_2)L -\beta_0\Gamma_1\frac{\zeta_2}{4}-\zeta_2\beta_1+\frac{2\beta_0^2}{3}\left(\zeta_3-\frac{82}{9}\right)+26\beta_0C_A\left(\zeta_4-\frac{8}{27}\right)\Big]$$

We found 3-loop rapidity anomalous dimension

$$\mathcal{D}^{(3)}_{L=0}=C_A^2\left(\frac{297029}{1458}+\frac{88}{3}\zeta_2\zeta_3+\dots+96\zeta_5\right)+\dots+C_FN_f\left(\frac{-152}{9}\zeta_3-8\zeta_4+\frac{11711}{54}\right)$$

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$$\mathcal{D}^{(3)}_{L=0} = -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3 - \frac{88}{3}\zeta_2\zeta_3 - 192\zeta_5 - \frac{297029}{729} + \frac{6392}{81}\zeta_2 + \frac{154}{3}\zeta_4\right) - \frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3 + \frac{62626}{729} - \frac{824}{81}\zeta_2 + \frac{20}{3}\zeta_4\right) - \frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3 + \frac{1711}{27} - 16\zeta_4\right) - \frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3 - \frac{1856}{729}\right)$$

- Coincides with the one calculated directly [Li,Zhu,1604.01404]
- ~~Amazingly but~~ the logarithmic structure of rapidity anomalous dimension also restored

$$\mu^2 \frac{d}{d\mu^2}\mathcal{D}(a_s(\mu), \mathbf{b}) = \frac{\Gamma_{cusp}(a_s(\mu))}{2}$$

vs.

$$\nu^2 \frac{d}{d\nu^2}\gamma_s(\nu, v) = \frac{\Gamma_{cusp}}{2}$$

UV anomalous dimensions independent on ϵ. UV anomalous dimension of rapidity anomalous dimension also.

General matrix rapidity anomalous dimension

- The "matrix" soft anomalous dimension is know up to NNNLO (3-loop), where "quadrapole" terms appear [Almelid,Duhr,Gardi,1507.00047]
- Using the same trick we restore the rapidity anomalous dimension for the most general case

$$\mathbf{D}(\{b\}) = \overbrace{\sum_{1\leqslant i<j}^{N} \mathbf{T}_i^A\mathbf{T}_j^A \mathcal{D}(L_{ij},a_s)}^{\text{dipole = TMD}} +$$
$$\underbrace{f^{AB\alpha}f^{\alpha CD}\Big[\sum_{i,j,k,l}\mathbf{T}_i^A\mathbf{T}_j^B\mathbf{T}_k^C\mathbf{T}_l^D\tilde{\mathcal{F}}(b_i,b_j,b_k,b_l)+\sum_{\substack{j<k\\ i\neq j,k}}\{\mathbf{T}_i^A,\mathbf{T}_i^D\}\mathbf{T}_j^B\mathbf{T}_k^C\tilde{C}(b_i,b_j,b_k)\Big]}_{\text{quadrapole, starts at } a_s^3}+\ldots$$

$$\tilde{\mathcal{F}}(b_i,b_j,b_k,b_l) = a_s^3\mathcal{F}(\rho_{ikjl},\rho_{iljk})+\mathcal{O}(a_s^4), \qquad \tilde{C}(b_i,b_j,b_k) = a_s^3\left(-\zeta_2\zeta_3-\zeta_5/2\right)+\mathcal{O}(a_s^4),$$

- $\rho_{ijkl} = \frac{b_{ij}b_{kl}}{b_{ik}b_{jl}}$, quadrapole part is conformal invariant at LO.
- 2-loop result coincides with one presented here or in [AV,1608.04920]
- Probably would be never used practically....

Rapidity renormalization theorem (weak)

- Any matrix element build of operators $\mathbf{\Xi}$ and other finite operators, which do not overlap under conformal-stereographic projection, has non-overlapping rapidity divergences related to every $\mathbf{\Xi}$.
- The non-overlapping rapidity divergences can be removed by multiplication of every $\mathbf{\Xi}$ by rapidity-renormalization matrix $\mathbf{R}$.

- Hold in conformal field theory and QCD
- Rapidity renormalization introduces rapidity scaling parameter ζ and R-RGE (CSS equation for TMD)
- Similar statement in TMD case (N=2), without proof, has been suggested in [Chiu,Jain,Neil,Rothstein,1202.0814]

Sketch of proof

- Theorem holds in conformal theory, due to conformal map of divergences.
- The power counting of rapidity divergences is independent on ϵ, they are "2D". $d = 2 + (d - 2)$.
- The relations between diagrams are also independent on ϵ, due to Ward identities.
- In QCD theorem holds at ϵ^*
- Thus, it hold at any ϵ.

Consequences (1)

Rapidity factorization

E.g. TMD factorization

$$\frac{d\sigma}{dQdyd^2q_t} \sim \int d^2b e^{-i(qb)} H(Q^2) \underbrace{\Phi_{h1}(z_1,b)\, R(\zeta_+)}_{\substack{F(z,b,\zeta_+)\\ \text{finite TMDPDF}}} \overbrace{R(\zeta_+) R^{-1}(\zeta_+)}^{=1} \underbrace{R^{-1}(\zeta_+)\, S(b)\, R(\zeta_-)}_{\substack{\varrho(\zeta_+,\zeta_-)\\ \text{finite}}} \overbrace{R(\zeta_-) R^{-1}(\zeta_-)}^{=1} \underbrace{R^{-1}(\zeta_-)\, \Delta_{h2}(z_2,b)}_{\substack{D(z,b,\zeta_-)\\ \text{finite TMDFF}}}$$

- There is rapidity-renormalization scheme dependence.
- The common choice $\varrho(\zeta_+,\zeta_-) = 1$ ($R = \exp(-A/2 \ln\delta - B/2)$ see slide 8)

E.g. double-Drell-Yan

$$\frac{d\sigma}{dX} \sim \int H_1(Q_1^2) H_2(Q_2^2) \underbrace{\mathbf{F}_{h1}(z_1,b)\, \mathbf{R}(\zeta_+)}_{\substack{\mathbf{F}(z,b,\zeta_+)\\ \text{finite DPD}}} \overbrace{\mathbf{R}(\zeta_+)\mathbf{R}^{-1}(\zeta_+)}^{=1} \underbrace{\mathbf{R}^{-1}(\zeta_+)\, \mathbf{\Sigma}(b)\, \mathbf{R}(\zeta_-)}_{\substack{\varrho(\zeta_+,\zeta_-)\\ \text{finite}}} \overbrace{\mathbf{R}(\zeta_-)\mathbf{R}^{-1}(\zeta_-)}^{=1} \underbrace{\mathbf{R}^{-1}(\zeta_-)\, \bar{\mathbf{F}}_{h2}(z_2,b)}_{\substack{\bar{\mathbf{F}}(z,b,\zeta_-)\\ \text{finite DPD}}}$$

- It is possible to choose $\varrho(\zeta_+,\zeta_-) = 1$ at NNLO

Consequences (2)

Summation prescriptions

- The non-perturbative part associated with renormalons can be also factorized. [Scimemi,AV,1609.06047]
- "Renormalon evaporation"

$$\mathcal{D} \text{ has renormalon singularities} \longrightarrow \mathcal{D}(\epsilon^*) \text{ has not renormalon singularities}$$

- Can be used as a summation prescription.

Absence of "naive" TMD factorization at higher orders

- TMDs of higher *dynamical twist*,e.g.

$$e(x,\mathbf{b}) \simeq \int d\xi\ \bar{q}_i(\xi+\mathbf{b})...q_i(0)$$

have overlapping rapidity singularities (from the "small" component of quark field).

- Can not be removed by the same procedure, (or soft factor) [I.Scimemi,AV,to be publ.]
- Either non-factorizable, either factorization has different form (several soft factors?)

Many others! (not studied)

Conclusion

- The rapidity divergences are alike UV divergences
- Both can be connected within the conformal field theory
- Using the fact that QCD restores conformal invariance at ϵ^* we can match conformal statement to QCD order by order

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- It can be checked up to three-loop order (!) for N=2 case (TMD)
- It can be checked up to two-loop order for general case
- Leads to prediction of three-loop general rapidity anomalous dimension

$$\boldsymbol{\Xi}^{rap.finite}(\{\mathbf{b}\}) = \boldsymbol{\Xi}(\{\mathbf{b}\})\mathbf{R}(\{\mathbf{b}\})$$

- Rapidity renormalization theorem (in weak form) is formulated
- Multiple consequences!

Universität Regensburg