Transverse momentum dependent distributions theory and practice

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Regensburg
May 2017

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Rapidity divergence renormalization theorem theory and practice

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Rapidity divergences are essential part of TMD factorization (SCET II)

- Up to recent time it was only assumed that rapidity divergences can be factorized. See e.g. Collins' book,

> **13.7 (****) Complete the proofs of all the results in this chapter, notably those concerning the application of the subtraction formalism to processes in a non-abelian gauge theory with TMD functions, and the expression of these functions in terms of operator matrix elements with Wilson lines.**
>
> **13.11 (***) The final definition of the TMD fragmentation function (13.42) involves a product of an unsubtracted fragmentation function and several Wilson-line factors. If possible, express Feynman graphs for this quantity as graphs for the unsubtracted fragmentation function with a systematic subtraction procedure applied. Again, publish the result if you are the first to solve this problem.**

- TMD factorization is the simplest case. The same problem arise in, say, multi-Drell-Yan scattering or hadron-to-hadron production.

Outline of talk

- State of problem: soft factors and rapidity divergences
- Example 1 &2 : Soft factors and rapidity decomposition for TMD factorization, and for Double-Drell-Yan ([AV,1608.04920])
- Soft/rapidity correspondence and its consequences ([AV,1610.05791])
- Rapidity divergence renormalization theorem and its consequences.

General structure of the factorization theorems

The modern factorization theorems have the following general structure

$$\underbrace{\frac{d\sigma}{dX}}_{cross-X} = \underbrace{H}_{\substack{\text{Hard part}\\\text{perturbative}}} \times \underbrace{f_1 \otimes ... \otimes J_2}_{\substack{\text{Parton distributions}\\\text{jet-functions, etc}\\\text{Non-pertrubative}\\\text{universal}}} \times \underbrace{S}_{\substack{\text{Soft factor(s)}\\\text{perturbative ?}}} + \text{Some power suppressed terms}$$

- This is typical outcome of SCET
- For many interesting cases the individual terms in the product are singular, and requires redefinition/refactorization
- In general, the factorization task is hidden in soft factors, (they mix the singularities of different field modes)

TMD factorization for SIDIS ($h_1+\gamma^*\to h_2+X$)

TMD factorization ($Q^2\gg q_T^2$) gives us the following expression

$$\frac{d\sigma}{dQdy\,d^2q_T}\sim\int d^4x\,e^{iqx}\sum_X\langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

TMD factorization

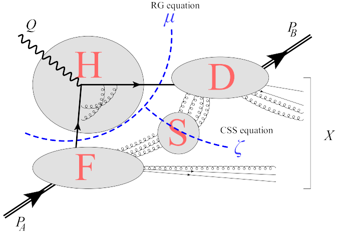
$$\frac{d\sigma}{dQdy\,d^2q_T}\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



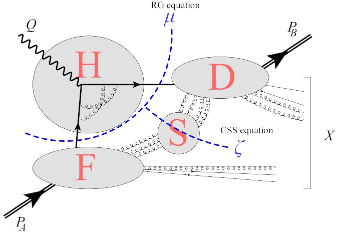
All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

In the TMD case : $S(b;\delta^+\delta^-)=\sqrt{S(b;\delta^+\zeta^+)}\sqrt{S(b;\delta^-\zeta^-)}$ ~~NOT PROVED!~~ PROVED!

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities
$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;F(z_1,b_T;\zeta^+)\;D(z_2,b_T;\zeta^-)+Y$$



TMD PDF
$\sqrt{S}\Phi_{h_1}$
(regular)

TMD FF
$\sqrt{S}\Delta_{h_2}$
(regular)

Many statements on TMDs are made in the assumption of rapidity factorization.

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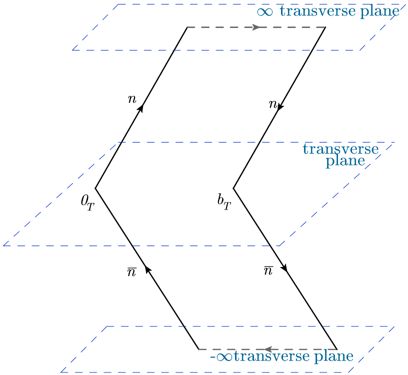
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n \cdot \bar{n}) = 1$$

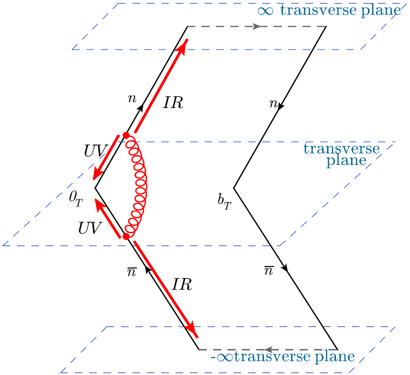
Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Multiple divergences!

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$

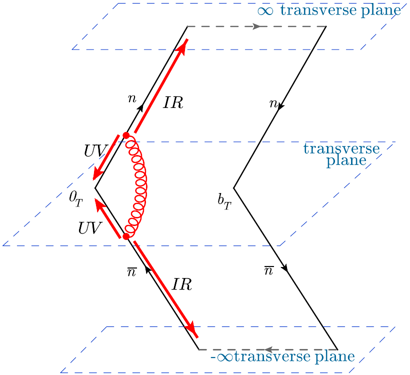


$$\begin{aligned}\int \quad dxdy \quad D(x-y) &= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-} \\ &= \int_0^\infty \frac{dx^+}{x^+} \int_0^\infty \frac{dy^-}{y^-} \\ &= (\mathrm{UV}+\mathrm{IR})(\mathrm{UV}+\mathrm{IR})\end{aligned}$$

Some people set it to zero, in dim.reg.

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$

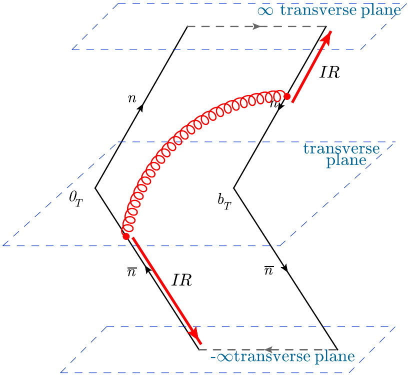


$$\begin{aligned}\int \quad dxdy \quad & D_{\mathrm{dim.reg}}(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(x^+y^-)^{1-2\epsilon}} \\ = \quad & \int_0^\infty \frac{dx^+}{(x^+)^{1-2\epsilon}} \int_0^\infty \frac{dy^-}{(y^-)^{1-2\epsilon}} \\ = \quad & \left(0^\epsilon + 0^{-\epsilon}\right)\left(0^\epsilon + 0^{-\epsilon}\right)\end{aligned}$$

$\epsilon > 0$ for one term, $\epsilon < 0$ for another.
Very dirty trick.

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \text{IR at } x, y \to \infty$$

However, it exactly cancels IR from the previous diagram

Proved at all orders,

e.g.[Echevarria,Scimemi,AV,1511.05590]

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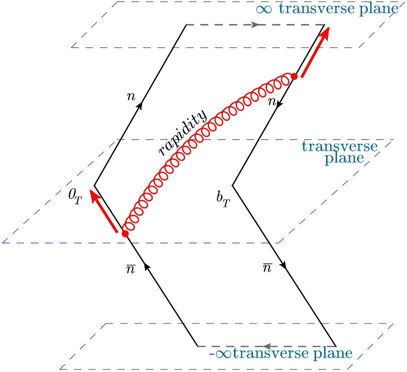
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned} \int \quad dxdy \quad & D(x-y) \\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)} \\ = \quad & \text{rap. div. at } \lim_{\lambda\to 0}\{x=\lambda, y=\lambda^{-1}\} \end{aligned}$$

Rapidity divergence is a special kind of divergences, UV& IR

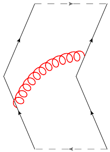
Does not cancel.

Does not regularized by dim.reg.

δ-regularization + dimension regularization($\epsilon > 0$)

$$P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)e^{-\delta\sigma}\right)$$

Nice and convenient composition of regularizations, that clear separates divergences.



$$= \quad \int_0^\infty dx^+ \int_0^\infty dy^- \frac{e^{-\delta^+ y^-} e^{-\delta^- x^+}}{(2x^+y^- + \mathbf{b}_T^2)^{1-\epsilon}}$$

$x^+ \to zL, \quad y^- \to L/z$

In this calculation scheme every divergece takes particular form

$$\left(\frac{\mathbf{b}^2}{4}\right)^\epsilon \left(\ln\left(\delta^+\delta^- \frac{\mathbf{b}^2 e^{2\gamma_E}}{4}\right) - \psi(-\epsilon) - \gamma_E\right) + \left(\delta^+\delta^-\right)^{-\epsilon}\Gamma^2(-\epsilon)$$

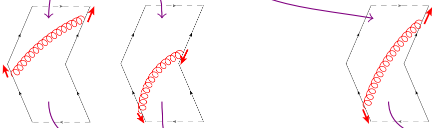
δ-regularization + dimension regularization($\epsilon > 0$)

$$P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma) e^{-\delta\sigma}\right)$$

Nice and convenient composition of regularizations, that clear separates divergences.

$x^+ \to zL, \quad y^- \to L/z$

$$\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{2LdL}{(L^2+\mathbf{b}^2)^{1-\epsilon}} e^{-L(z\delta^+ + \delta^-/z)}$$



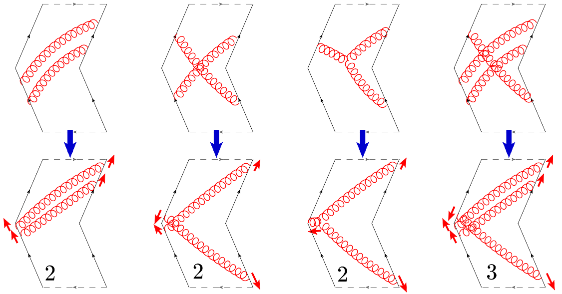
$$\left(\frac{\mathbf{b}^2}{4}\right)^\epsilon \left(\ln\left(\delta^-\delta^+\frac{\mathbf{b}^2 e^{2\gamma_E}}{4}\right) - \psi(-\epsilon) - \gamma_E\right) + \left(\delta^+\delta^-\right)^{-\epsilon}\Gamma^2(-\epsilon)$$

- Rapidity divergences happen only in one sector of diagram and independent on another
- The rules of counting the rapidity logarithms $\ln(\delta^+\delta^-)$ are very simple.

- Sub-graph contains rapidity divergence if gluon can be radiated from cusp to infinity.
- Count all divergent sub-graphs and remove them
- Repeat

Just like like UV divergences! (not accidental!)

Examples of high degree divergences

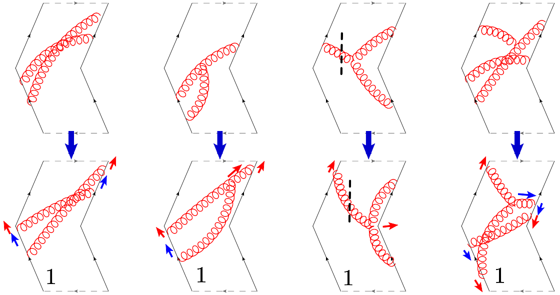


$\ln^2 \delta^+\delta^-$ $\quad$ $\ln^2 \delta^+\delta^-$ $\quad$ $\ln^2 \delta^+\delta^-$ $\quad$ $\ln^3 \delta^+\delta^-$

- Sub-graph contains rapidity divergence if gluon can be radiated from cusp to infinity.
- Count all divergent sub-graphs and remove them
- Repeat

Just like like UV divergences! (not accidental!)

Examples of low degree divergences



$\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$ $\ln \delta^+ \delta^-$

Typical expression ($\boldsymbol{\delta} = \delta^+\delta^-$)

Generally (say at NNLO) one expects the following form (finite ϵ, $\delta \to 0$)

$$S^{[2]} = \underbrace{\overbrace{A_1\boldsymbol{\delta}^{-2\epsilon} + A_2\boldsymbol{\delta}^{-\epsilon}\boldsymbol{B}^{\epsilon}}^{\text{IR}} + \boldsymbol{B}^{2\epsilon}\Big(A_3\ln^2(\boldsymbol{\delta}B)}_{\text{cancel in sum of diagram}} + A_4\ln(\boldsymbol{\delta}B) + A_5\Big)$$

- Terms $\sim \boldsymbol{\delta}^{-\epsilon}$ cancel exactly at all orders (simple to prove from $S_{b\to 0} \equiv 1$)
- A_3 cancels due to Ward identity (alike leading UV pole for cusp)(what about NNNLO?)

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$ (at finite ϵ)

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

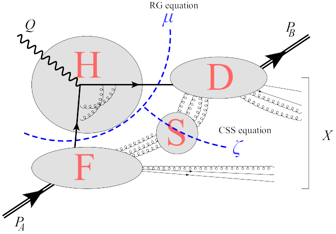
It allows to split rapidity divergences and define individual TMDs.

$$\exp(A\ln(\delta^+\delta^-)+B)=\exp\left(\frac{A}{2}\ln((\delta^+)^2\zeta)+\frac{B}{2}\right)\exp\left(\frac{A}{2}\ln((\delta^-)^2\zeta^{-1})+\frac{B}{2}\right)$$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\ \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities
$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\ F(z_1,b_T;\zeta^+)\ D(z_2,b_T;\zeta^-)+Y$$



TMD PDF
$\sqrt{S}\Phi_{h_1}$
(regular)

TMD FF
$\sqrt{S}\Delta_{h_2}$
(regular)

The extra "factorization" introduces extra scale ζ.
And corresponded evolution equation

$$\zeta\frac{d}{d\zeta}F=\frac{A}{2}F=-\mathcal{D}F$$

Rapidity anomalous dimension

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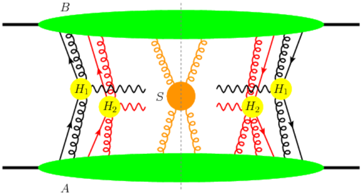
- Checked at NNLO (two-loops) [Echevarria,Scimemi,AV,1511.05590],[Lübbert,Oredsson,Stahlhofen,1602.01829], [Li,Neill,Zhu,1604.00392]
- The direct all-order proof is absent (but will be presented here.).
- Nothing is known about large (finite) b (however, leading renormalon part also factorizes [Scimemi,AV,1609.06047]).
- SIDIS is not the only process where this problem appear (but it is the simplest one).

Let me present another, less practical, but more general example.

Double-Drell-Yan scattering



pictures from [1510.08696]

Double Drell-Yan scattering

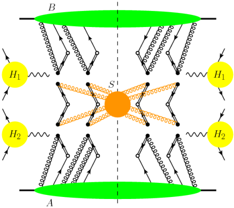
- Experimental status is doubtful
- (Collinear/Glauber) Factorization is proved [Diehl,et al,1510.08696]
- The same SCET II factorization, as in SIDIS.
- The same problem of rapidity factorization

DPD soft factor
(very singular)

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\; F^A_{h1}(z_{1,2}, b_{1,2,3,4}) S^{AB}(b_{1,2,3,4}) \bar{F}^B_{h2}(z_{1,2}, b_{1,2,3,4}) + Y$$

power suppressed terms

DPD (singular)

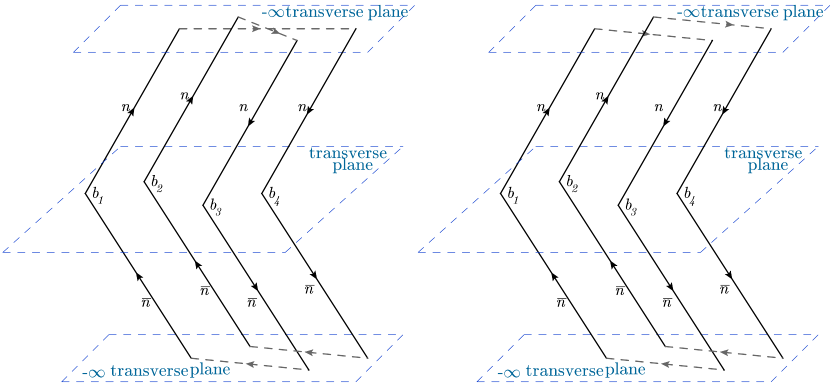


Structure is similar to TMD Drell-Yan
but now it contains
COLOR

Color structure makes a lot of difference

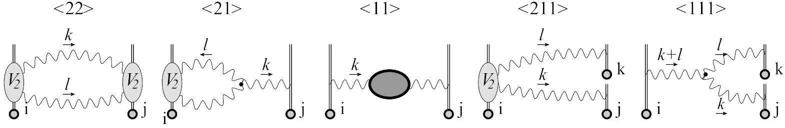
$$F^{A}_{h1} S^{AB} \bar{F}^{B}_{h2} \xrightarrow{\text{singlets}} (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix}$$

- Soft-factors $S^{\mathbf{ij}}$ are sum of Wilson loops and double Wilson loops (all possible connections).
- Soft-factors non-zero even in the integrated case.



Evaluation at NNLO [AV,1608.04920]

- Brute force evaluation would lead a lot of (similar) diagrams.
- The better way is to compute the generating function [AV, 1406.6253, 1501.03316]



- All non-trivial three-Wilson line interactions cancel.
- At (up to) NNLO Double-SF expresses via TMD soft factor only!

$$\text{TMD SF} \quad : \quad \ln S^{\text{TMD}} = \sigma(\mathbf{b})$$

$$\text{Single loop SF} \quad : \quad \ln S^{[4]} = \sigma(\mathbf{b}_{12}) - \sigma(\mathbf{b}_{13}) + \sigma(\mathbf{b}_{14}) + \sigma(\mathbf{b}_{23}) - \sigma(\mathbf{b}_{24}) + \sigma(\mathbf{b}_{34})$$
$$+ \frac{C_A}{4C_F}(\sigma(\mathbf{b}_{13}) - \sigma(\mathbf{b}_{14}) - \sigma(\mathbf{b}_{23}) + \sigma(\mathbf{b}_{24}))(\sigma(\mathbf{b}_{12}) - \sigma(\mathbf{b}_{13}) - \sigma(\mathbf{b}_{24}) + \sigma(\mathbf{b}_{34}))$$

$$\text{Double loop SF} \quad : \quad \ln S^{[1]} = \sigma(\mathbf{b}_{14}) + \sigma(\mathbf{b}_{23}) + \frac{1}{2}\left(\frac{C_A}{4C_F} - 1\right)(\sigma(\mathbf{b}_{12}) - \sigma(\mathbf{b}_{13}) - \sigma(\mathbf{b}_{24}) + \sigma(\mathbf{b}_{34}))^2$$

- It is exact statement resulting from the algebra of fields, and independent on regularization procedure.

REMINDER: TMD factorization

$$S^{\text{TMD}} = e^{\sigma(\mathbf{b})} = e^{\sigma^+(\mathbf{b})}e^{\sigma^-(\mathbf{b})}, \qquad \sigma^{\pm} = \frac{A}{2}\ln((\delta^+)^2\zeta^{\pm 1}) + \frac{B}{2}$$

Matrix factorization of rapidity divergences

Using the decomposition above, inserting it into DPD SF we obtain matrix relation

$$S^{\text{DPD}} = s^T(\ln(\delta^+)) \cdot s(\ln(\delta^-))$$

$$s = \exp\left[\begin{pmatrix} A^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ A^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix}\ln(\delta) + \begin{pmatrix} B^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ B^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix}\right]$$

$A^{\mathbf{ij}}$ and $B^{\mathbf{ij}}$ are rather complicated non-linear compositions of TMD's A and B

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Finalizing DPD factorization

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2) \;(F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix} + Y$$

spliting rapidity singularities
$S(b_{1,2,3,4}) \to s^T(b_{1,2,3,4}; \zeta^+) s(b_{1,2,3,4}; \zeta^-)$

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}]\, e^{-i(qb)_T} H_1(Q_1^2) H_2(Q_2^2)\; F(z_{1,2}, b_{1,2,3,4}; \zeta^+) \bar{F}(z_{1,2}, b_{1,2,3,4}; \zeta^-) + Y$$

DPD
$(sF_{h_1})^T$
(regular)

DPD
sF_{h_2}
(regular)

Matrix rapidity evolution

$$\frac{dF(z_{1,2}, \mathbf{b}_{1,2,3,4}; \zeta, \mu)}{d\ln\zeta} = -F(z_{1,2}, \mathbf{b}_{1,2,3,4}; \zeta, \mu)\mathbf{D}(\mathbf{b}_{1,2,3,4}, \mu)$$

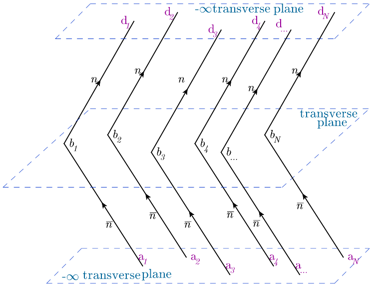
where $\mathbf{D}$ is matrix build (linearly) of TMD rapidity anomalous dimensions. E.g.

$$D^{\mathbf{18}} = \frac{\mathcal{D}(\mathbf{b}_{12}) - \mathcal{D}(\mathbf{b}_{13}) - \mathcal{D}(\mathbf{b}_{24}) + \mathcal{D}(\mathbf{b}_{34})}{\sqrt{N_c^2 - 1}}$$

Let's look at multi-parton scattering

- Just as double-parton, but multi..(four WL's $\to$ arbitrary number WL's)
- (Collinear/Glauber)Factorization can be/is proven [Diehl, Schäfer,et al,1510.08696,1111.0910]
- Too many color-singlets, better to work with explicit color indices (color-multi-matrix)

$$\Sigma^{(a_1...a_N);(d_1...d_N)}(\mathbf{b}_1,...,\mathbf{b}_N)=\mathbf{\Sigma}(\mathbf{b}_{1,..,N})$$



Result at NNLO is amazingly simple

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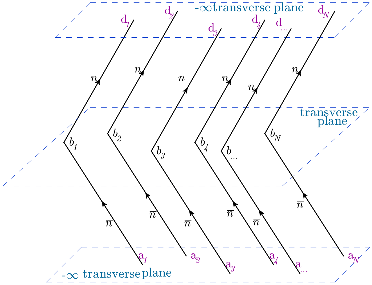

Result at NNLO is amazingly simple

$$\mathbf{\Sigma}(\mathbf{b}_{1,..,N})=\exp\left(-\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A\sigma(\mathbf{b}_{ij})+\mathcal{O}(a_s^3)\right)$$

- $\mathbf{T}_i^A\mathbf{T}_j^A$ = "color-dipole"
- $\mathcal{O}(a_s^3)$ contains also "color-multipole" terms
- Rapidity factorization for dipole part is straightforward (assuming TMD factorization)

Main question

- Can rapidity factorization be proven at all orders?

Main lessons

- Rapidity divergences characterize interaction of origin with infinity. Do not depend on the rest!
- Has very similar (topologically) structure with UV divergences.

Hint

- The color dependent expression reminds us something.... Multi-particle production!

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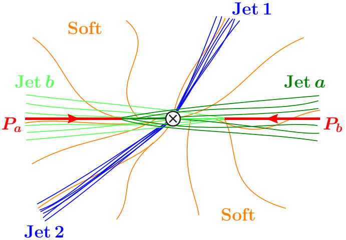
END OF INTRODUCTION

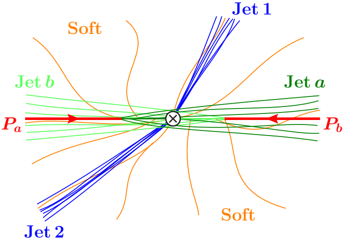
Multi-jet production



picture from
[Stewart,Tackmann,Waalewijn,0910.0467]

$$\frac{d\sigma}{dX} = H^{IJ} f_A \otimes f_B \otimes J_1 \otimes J_2\, S^{JI}$$

Multi-jet production



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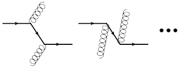
Hard part: pertrubative



Parton distributions
$f \sim \langle h|\bar{q}[\text{Wilson lines}]q|h\rangle$

$$\frac{d\sigma}{dX} = H^{IJ}\,\overbrace{f_A \otimes f_B}\otimes \underbrace{J_1 \otimes J_2}\,\mathcal{S}^{JI}$$

Jet functions
$J \sim \langle 0|q|X_J\rangle\langle X_J|\bar{q}|0\rangle$

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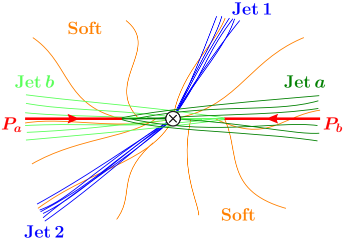
Multi-jet production



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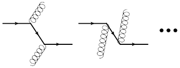

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Jet functions
$J \sim \langle 0|q|X_J\rangle\langle X_J|\bar q|0\rangle$

Soft factor

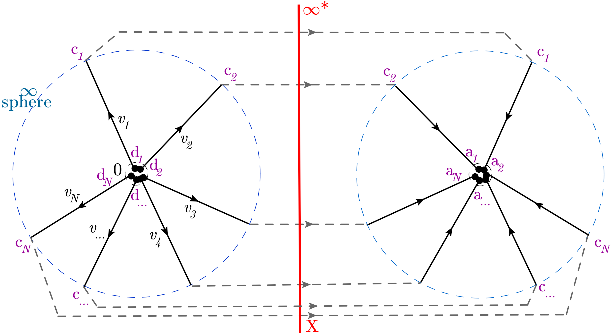
- The details of definition for soft factors differs from process to process. Generally: $S \sim \langle 0|[[\text{Wilson lines}]|0\rangle$
- The soft factor can be product of soft factors
- Soft factors also color matrix (but coupled to hard part).

Structure of N-jet soft factor

$$\mathcal{S}^{\{ad\}}(\{v\}) = \sum_X w_X \Pi_X^{\dagger\{ac\}}(\{v\})\Pi_X^{\{cd\}}(\{v\}) \quad \text{Soft factor}$$

$$\Pi_X^{\{cd\}}(\{v\}) = \langle X|T[\Phi_{v_1}^{c_1 d_1}(0)...\Phi_{v_N}^{c_N d_N}(0)]|0\rangle$$

$$\text{Wilson "ray"} \quad \mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma+x)\mathbf{T}^A\right)$$



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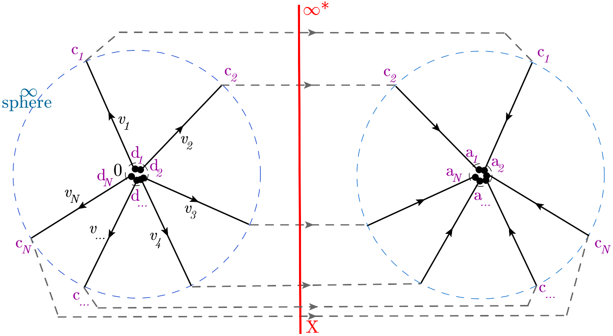
Structure of generic soft factor

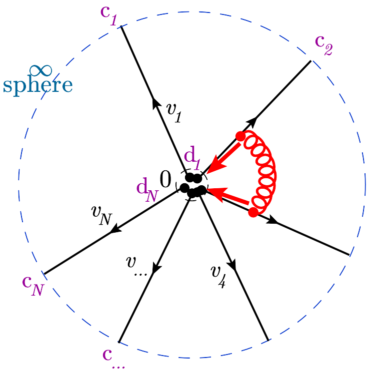
- Depending on w_X soft factors can have very different structure
- Such structure of soft factor appears in many places: multi-jet product, event shapes, hard-collinear factorization and Sudakov factorization, threshold resummation.
- In the most popular configurations the soft factor (or its parts) have been evaluated up to N^{3}LO
- In the following I consider only $v_i^2 = 0$ (massless partons)



Soft anomalous dimension

- With the proper choice of w_X the multi-jet soft factor is IR finite.
- UV divergences are renormalized by the appropriate matrix

$$\mathbf{S}^{\text{ren}}(\{v\}) = \mathbf{Z}^{\dagger}(\{v\})\mathbf{S}(\{v\})\mathbf{Z}(\{v\})$$



Renormalization factor knows only about $\mathbf{\Pi}$

$$\mathbf{S}^{\text{ren}}(\{v\}) = \sum_X w_X \mathbf{\Pi}_X^{\dagger\text{ren}}(\{v\})\mathbf{\Pi}_X^{\text{ren}}(\{v\})$$

$$\mathbf{\Pi}_X^{\text{ren}}(\{v\}) = \mathbf{\Pi}_X^{\text{ren}}(\{v\})\mathbf{Z}(\{v\})$$

- Individually $\mathbf{\Pi}$ is a horrible object
  - Not gauge invariant
  - IR singular
  - Dependent on X
- However UV factor $\mathbf{Z}$ is well-defined
  - Gauge invariant
  - Independent on X

- The RG anomalous dimension for operator $\mathbf{\Pi}$ is called "soft anomalous dimension"

$$\mu^2 \frac{d}{d\mu^2} \mathbf{\Pi}(\{v\}) = \mathbf{\Pi}(\{v\}) \boldsymbol{\gamma}_s(\{v\})$$

The soft anomalous dimension is subject of intensive studies

- It is known up to 3-loops (4-loops in $\mathcal{N}=4$ SYM)
- It is conformally invariant (rescaling of the vectors $\{v\}$)
- It has structure (dipole part)

$$\underbrace{\boldsymbol{\gamma}_s(\{v\})}_{\substack{\text{sAD for}\\ \text{multi-jet}}} = -\sum_{i<j} \mathbf{T}_i^A \mathbf{T}_j^A \underbrace{\tilde{\gamma}(v_i \cdot v_j)}_{\substack{\text{sAD for}\\ \text{double-jet}}} + \ldots$$

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Compare with the rapidity anomalous dimension (dipole part)

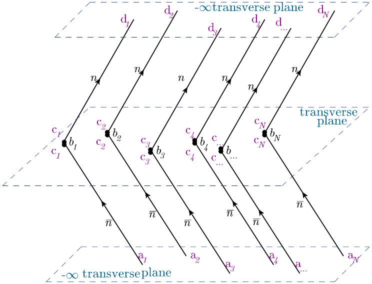
$$\underbrace{\mathbf{D}(\{b\})}_{\substack{\text{rAD for}\\ \text{multi-PD}}} = -\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A \underbrace{\mathcal{D}((b_i-b_j)^2)}_{\substack{\text{rAD for}\\ \text{TMD}}} + ...$$

Rewriting multi-parton scattering soft factor

$$\Sigma^{\{ad\}}(\{\mathbf{b}\}) = \langle 0|T[\left(\Phi_{-\bar{n}}^{a_1c_1}\Phi_{-n}^{\dagger c_1 d_1}\right)(\mathbf{b}_1)...\left(\Phi_{-\bar{n}}^{a_Nc_N}\Phi_{-n}^{\dagger c_N d_N}\right)(\mathbf{b}_N)]|0\rangle$$ MPS Soft factor

$$\mathbf{\Sigma}(\{\mathbf{b}\}) = \sum_X \tilde{w}_X \mathbf{\Xi}^\dagger_{\bar{n},X}(\{\mathbf{b}\})\mathbf{\Xi}_{n,X}(\{\mathbf{b}\})$$

$$\Xi_{n,X}^{\{cd\}}(\{\mathbf{b}\}) = \langle X|T[\Phi_{-n}^{\dagger c_1 d_1}(\mathbf{b}_1)...\Phi_{n}^{\dagger c_N d_N}(\mathbf{b}_N)]|0\rangle$$



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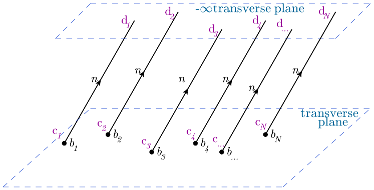


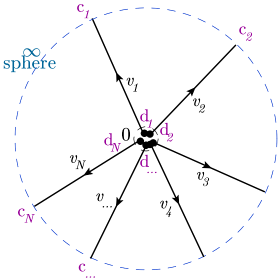
Unnatural act

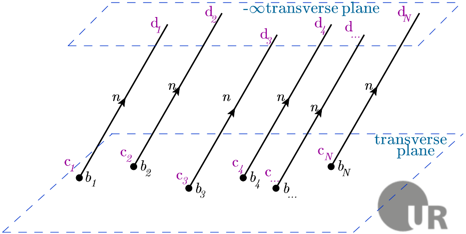
In many aspects $\boldsymbol{\Xi}$ is similar to $\boldsymbol{\Pi}$

- Individually $\boldsymbol{\Xi}$ is a horrible object
  - Not gauge invariant
  - IR singular
  - End-point singularities
  - Dependent on X
- However it is as horrible as $\boldsymbol{\Pi}$

Conformal-Stereographic transformation

$$\mathcal{C}: \quad \{x^+, x^-, \mathbf{x}_T\} \longrightarrow \left\{-\frac{1}{2x^+}, x^- - \frac{\mathbf{x}_T^2}{2x^+}, \frac{\mathbf{x}_T}{\sqrt{2}x^+}\right\}$$





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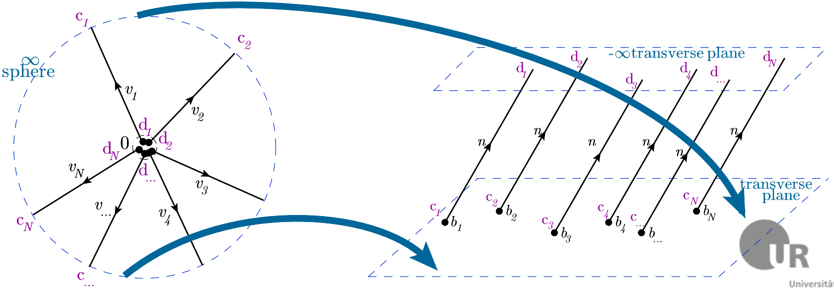
- ∞-sphere transforms to the transverse plane (at origin)



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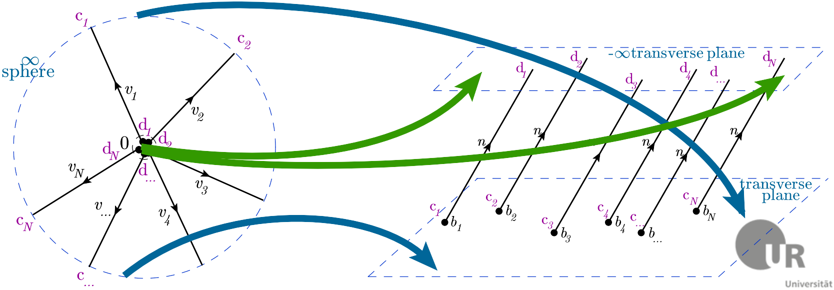

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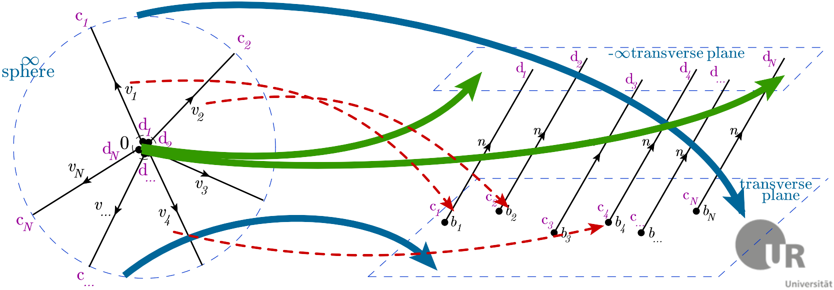

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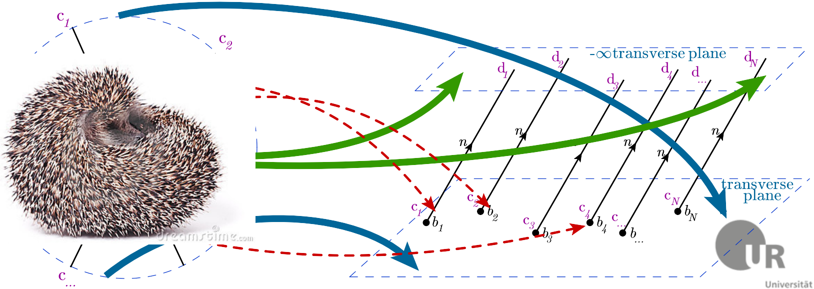

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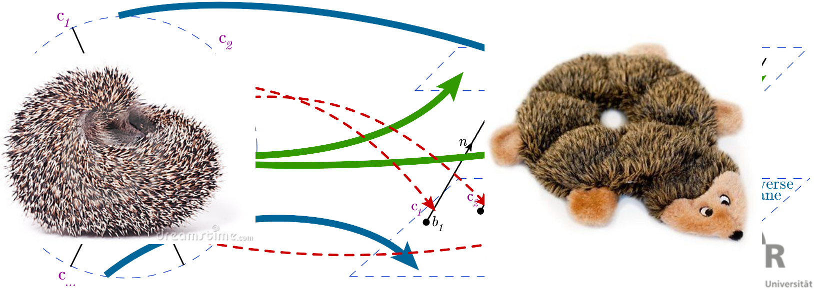

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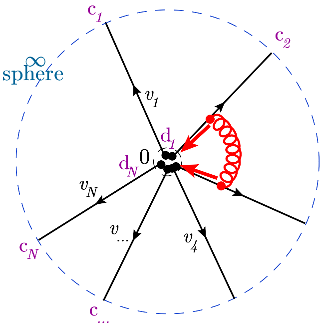
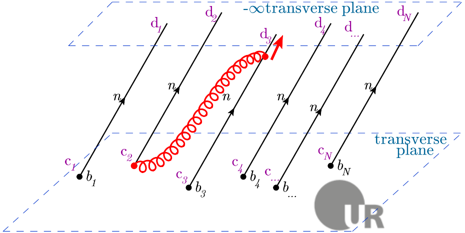
Conformal-Stereographic transformation

$$\mathcal{C}\Phi_v^{cd}(0) \;=\; \Phi_{-n}^{\dagger cd}\left(\frac{\mathbf{v}_T}{\sqrt{2}v^+}\right)$$

$$\mathbf{\Pi}_{X=0}\left(\{v\}\right) \longrightarrow \mathbf{\Xi}_{X=0}\left(\{\mathbf{b}\}\right)$$

- UV divergences of $\mathbf{\Pi}$ map onto the rapidity divergences $\mathbf{\Xi}$
- Rapidity divergent part of $\mathbf{\Xi}$ is gauge invariant and independent on X (like UV of $\mathbf{\Pi}$)





In conformal field theory

$$\mathbf{\Pi}_X(\{v\})\mathbf{Z}(\{v\}) \xrightarrow{\mathcal{C}} \mathbf{\Xi}_X(\{\mathbf{b}\})\mathbf{R}(\{\mathbf{b}\})$$

- In conformal field theory rapidity divergences renormalizable (for every factor $\mathbf{\Xi}$).
- $\mathbf{R}$ can be obtained from $\mathbf{Z}$ (by some transformation of regularizations)

$$\text{UV RGE:}\quad \mu^2\frac{d}{d\mu^2}\mathbf{\Pi}_X = \mathbf{\Pi}_X\boldsymbol{\gamma}_s \xrightarrow{\mathcal{C}} \zeta\frac{d}{d\zeta}\mathbf{\Xi}_X = 2\mathbf{\Xi}_X\mathbf{D}\quad \text{:Rap. RGE}$$

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Since anomalous dimensions independent on regularization we have simple relation

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\})$$

Indeed, such equality has been recently observed at NNLO in $\mathcal{N}=4$ SYM [Li,Zhu,1604.01404]

QCD at critical coupling

$\boxed{\text{Conformal symmetry of QCD is restored at critical coupling } a^*}$

In dimensional regularization

At critical coupling β-function vanish ($a_s = g^2/(4\pi)^2$)

$$\beta(g) = g\left(-\epsilon - a_s\beta_0 - a_s^2\beta_1 - ...\right),$$

Equation $\beta(g^*) = 0$ defines the value of $a_s^*(\epsilon)$ or equivalently defines the number of dimensions in which QCD critical

$$\beta(\epsilon^*) = 0 \qquad \longrightarrow \qquad \epsilon^* = -a_s\beta_0 - a_s^2\beta_1 - ...$$

In MS-like schemes UV anomalous dimension is independent on definition of ϵ.

$\boxed{\text{UV anomalous dimensions are conformal invariant}}$

[Vasiliev; 90's]
for modern applications see e.g.
[Braun,Manashov,1306,5644]

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Soft/rapidity anomalous dimension correspondence

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

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In QCD

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- Exact relation!
- Connects different regimes of QCD
- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$

→ Lets test it.

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TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\boldsymbol{B}=\frac{b^2}{4}$, $L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow\ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$ NLO | NLO $-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$$\underbrace{\gamma_s(\{v\})}_{a_s\gamma_s^{(1)}}=\underbrace{2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)}_{a_s\mathcal{D}^{(1)}}$$

Expand in a_s

$$a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{\nu^2}\right)+0\right)=a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{4e^{-2\gamma_E}}\right)+0\right)$$

Obvious relation, QCD is confirmal at leading order.

$$\nu^2=4e^{-2\gamma_E}$$

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$\gamma_s^{(1)}=L+0$

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+...)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$

NLO | NLO

N^{2}LO

$-2\left(\boldsymbol{B}^{\epsilon}\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$$\underbrace{a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}}\rightarrow\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)\leftarrow\underbrace{a_s\mathcal{D}^{(1)}+a_s^2?\mathcal{D}^{(2)}}$$

Expand in a_s

$$..+a_s^2\left(\Gamma_2L+\gamma^{(2)}\right)=..+2a_s^2\left(\mathcal{D}^{(2)}-2\beta_0(L^2+\zeta_2)\right)$$

We found 2-loop rapidity anomalous dimension

$$\mathcal{D}^{(2)}_{L=0}=\left(\frac{404}{27}-14\zeta_3\right)C_A-\frac{112}{27}\frac{N_f}{2}$$

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NLO | NLO

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N^{2}LO | N^{2}LO

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+\ldots)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$

$\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)$
$+\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\left(\frac{3(4-3\epsilon)}{2\epsilon}C_A-N_f\right)\Big)$
$+\boldsymbol{B}^{\epsilon}\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0+\frac{\beta_0}{2\epsilon^2}-\frac{\Gamma_1}{2\epsilon}=\mathcal{D}^{(2)}$

[Echevarria,Scimemi,AV,1511.05590]

N^{3}LO

$\gamma_s^{(3)}=[\frac{245}{3}C_A^2+\ldots]L+$
$+(-192\zeta_5C_A^2+\ldots+\frac{2080}{729}N_f^2)$

$$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}+a_s^3\gamma_s^{(3)} \qquad\qquad a_s\mathcal{D}^{(1)}+a_s^2\mathcal{D}^{(2)}+a_s^3?\mathcal{D}^{(3)}$$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$..+a_s^3\left(\Gamma_3L+\gamma^{(3)}\right)=..+2a_s^2\Big[\mathcal{D}^{(3)}-\frac{2\beta_0^2}{3}L^3-\left(\frac{\beta_0\Gamma_1}{2}+\beta_1\right)L^2+\beta_0(\gamma_1-2\beta_0\zeta_2)L$$
$$-\beta_0\Gamma_1\frac{\zeta_2}{4}-\zeta_2\beta_1+\frac{2\beta_0^2}{3}\left(\zeta_3-\frac{82}{9}\right)+26\beta_0C_A\left(\zeta_4-\frac{8}{27}\right)\Big]$$

We found 3-loop rapidity anomalous dimension

$$\mathcal{D}^{(3)}_{L=0}=C_A^2\left(\frac{297029}{1458}+\frac{88}{3}\zeta_2\zeta_3+\ldots+96\zeta_5\right)+\ldots+C_FN_f\left(\frac{-152}{9}\zeta_3-8\zeta_4+\frac{11711}{54}\right)$$

$$
\begin{aligned}
\mathcal{D}^{(3)}_{L=0} \quad = \quad & -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3-\frac{88}{3}\zeta_2\zeta_3-192\zeta_5-\frac{297029}{729}+\frac{6392}{81}\zeta_2+\frac{154}{3}\zeta_4\right) \\
& -\frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3+\frac{62626}{729}-\frac{824}{81}\zeta_2+\frac{20}{3}\zeta_4\right)- \\
& \frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3+\frac{1711}{27}-16\zeta_4\right)-\frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3-\frac{1856}{729}\right)
\end{aligned}
$$

- Coincides with the one calculated directly [Li,Zhu,1604.01404]
- The logarithmic structure of rapidity anomalous dimension also restored

$$\mu^2\frac{d}{d\mu^2}\mathcal{D}(a_s(\mu),\mathbf{b})=\frac{\Gamma_{cusp}(a_s(\mu))}{2}$$

vs.

$$\nu^2\frac{d}{d\nu^2}\gamma_s(\nu,v)=\frac{\Gamma_{cusp}}{2}$$

UV anomalous dimensions independent on ϵ. UV anomalous dimension of rapidity anomalous dimension also.

General matrix rapidity anomalous dimension

- The "matrix" soft anomalous dimension is know up to NNNLO (3-loop), where "quadrapole" terms appear [Almelid,Duhr,Gardi,1507.00047]
- Using the same trick we restore the rapidity anomalous dimension for the most general case

$$\mathbf{D}(\{b\}) = \overbrace{\sum_{1\leqslant i<j}^{N} \mathbf{T}_i^A \mathbf{T}_j^A \mathcal{D}(L_{ij}, a_s)}^{\text{dipole = TMD}} +$$

$$\underbrace{f^{AB\alpha} f^{\alpha CD} \Big[\sum_{i,j,k,l} \mathbf{T}_i^A \mathbf{T}_j^B \mathbf{T}_k^C \mathbf{T}_l^D \tilde{\mathcal{F}}(b_i, b_j, b_k, b_l) + \sum_{\substack{j<k \\ i\neq j,k}} \{\mathbf{T}_i^A, \mathbf{T}_i^D\} \mathbf{T}_j^B \mathbf{T}_k^C \tilde{C}(b_i, b_j, b_k) \Big]}_{\text{quadrapole, starts at } a_s^3} + \ldots$$

$$\tilde{\mathcal{F}}(b_i, b_j, b_k, b_l) = a_s^3 \mathcal{F}(\rho_{ikjl}, \rho_{iljk}) + \mathcal{O}(a_s^4), \qquad \tilde{C}(b_i, b_j, b_k) = a_s^3 \left(-\zeta_2\zeta_3 - \zeta_5/2\right) + \mathcal{O}(a_s^4),$$

- $\rho_{ijkl} = \frac{b_{ij} b_{kl}}{b_{ik} b_{jl}}$, quadrapole part is conformal invariant at LO.
- 2-loop result coincides with one presented here or in [AV,1608.04920]
- Probably would be never used practically....

Rapidity renormalization theorem (weak)

- Any matrix element build of operators $\mathbf{\Xi}$ and other finite operators, which do not overlap under conformal-stereographic projection, has non-overlapping rapidity divergences related to every $\mathbf{\Xi}$.
- The non-overlapping rapidity divergences can be removed by multiplication of every $\mathbf{\Xi}$ by rapidity-renormalization matrix $\mathbf{R}$.

- Hold in conformal field theory and QCD
- Rapidity renormalization introduces rapidity scaling parameter ζ and R-RGE (CSS equation for TMD)
- Similar statement in TMD case (N=2), without proof, has been suggested in [Chiu,Jain,Neil,Rothstein,1202.0814]

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Sketch of proof

- Theorem holds in conformal theory, due to conformal map of divergences.
- The power counting of rapidity divergences is independent on ϵ, they are "2D".
- Power of rapidity divergence is not higher the perturbative order.
- In QCD theorem holds at ϵ^*. I.e. exist such $\mathbf{Z}$

$$\mathbf{\Xi}^{\text{ren.}}(\zeta,\epsilon^*)=\mathbf{Z}(a_s,\ln\frac{\delta}{\zeta},\epsilon^*)\Big[\mathbf{I}+\sum_{n,k}a_s^n\ln^k\delta\;\mathbf{B}_{nk}(\epsilon^*)\Big] \qquad (1)$$

- Make perturbative shift $\epsilon^*\to\epsilon^{**}=\epsilon^*+a_s\alpha$. Show that $\mathbf{Z}(\epsilon^{**})$ can be obtained from $\mathbf{Z}(\epsilon^*)$ and $\mathbf{B}$.
- Iterate.

Consequences (1)

Rapidity factorization

E.g. TMD factorization

$$\frac{d\sigma}{dQdyd^2q_t} \sim \int d^2b e^{-i(qb)} H(Q^2) \underbrace{\Phi_{h1}(z_1,b)\, R(\zeta_+)}_{\substack{F(z,b,\zeta_+)\\ \text{finite TMDPDF}}} \overbrace{\underbrace{R^{-1}(\zeta_+)\, S(b)\, R(\zeta_-)}_{\substack{\varrho(\zeta_+,\zeta_-)\\ \text{finite}}}}^{=1} \underbrace{R^{-1}(\zeta_-)\, \Delta_{h2}(z_2,b)}_{\substack{D(z,b,\zeta_-)\\ \text{finite TMDFF}}}$$

- There is rapidity-renormalization scheme dependence.
- The common choice $\varrho(\zeta_+,\zeta_-)=1$ ($R=\exp(-A/2\ln\delta - B/2)$ see slide 8)

E.g. double-Drell-Yan

$$\frac{d\sigma}{dX} \sim \int H_1(Q_1^2)H_2(Q_2^2) \underbrace{\mathbf{F}_{h1}(z_1,b)\, \mathbf{R}(\zeta_+)}_{\substack{\mathbf{F}(z,b,\zeta_+)\\ \text{finite DPD}}} \underbrace{\mathbf{R}^{-1}(\zeta_+)\, \mathbf{\Sigma}(b)\, \mathbf{R}(\zeta_-)}_{\substack{\varrho(\zeta_+,\zeta_-)\\ \text{finite}}} \underbrace{\mathbf{R}^{-1}(\zeta_-)\, \bar{\mathbf{F}}_{h2}(z_2,b)}_{\substack{\bar{\mathbf{F}}(z,b,\zeta_-)\\ \text{finite DPD}}}$$

- It is possible to choose $\varrho(\zeta_+,\zeta_-)=1$ at NNLO

Consequences (2)

Counter example no-go theorem for hadron-to-hadron production [Mulders,Rogers,1001.2977]

- The no-go theorem is based on the observation that (at NNLO) the singlet-singlet entry to a colored soft factor (for $h_1 + h_2 \to h_3 + h_4 + X$) does not factorized correctly.
- It is indeed so (see e.g. example of double-Drel-Yan). But the full matrix factorizes correctly.

Absence of "naive" TMD factorization at higher orders

- TMDs of higher *dynamical twist*,e.g.

$$e(x, \mathbf{b}) \simeq \int d\xi \; \bar{q}_i(\xi + \mathbf{b}) ... q_i(0)$$

  have overlapping rapidity singularities (from the "small" component of quark field).
- Can not be removed by the same procedure, (or soft factor). Show explicitly in [D.Gutierres-Reyes,I.Scimemi,AV,1702.06558]
- Either non-factorizable, either factorization has different form (several soft factors?)

A bit of practice

Are these high-orders important?

YES! The TMD factorization covers wide region of logarithms. Therefore, it requires higher-orders of perturbation theory to minimize the theoretical uncertainties.

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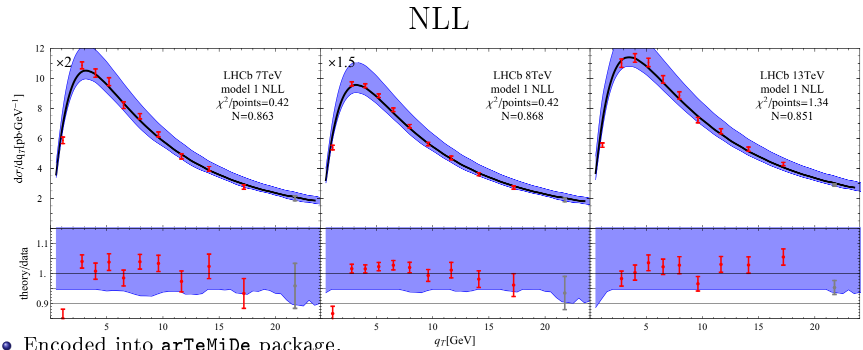

- Encoded into **arTeMiDe** package.
- The highest order contributions was included into the recent global fit of Drell-Yan data (E288+CDF+D0+ATLAS+CMS+LHCb) [I.Scimemi,AV,appear soon]
- The most even coverage of energy scales (for TMD business):$(4, 19.5)\text{GeV} < (Q, \sqrt{s}) < (150, 13000)\text{GeV}$

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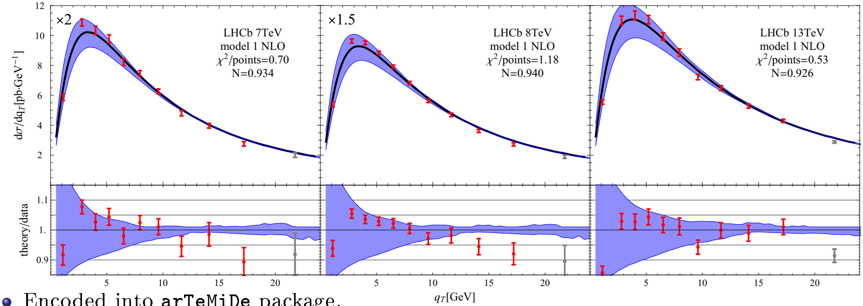
NLO(=standard for TMD community)



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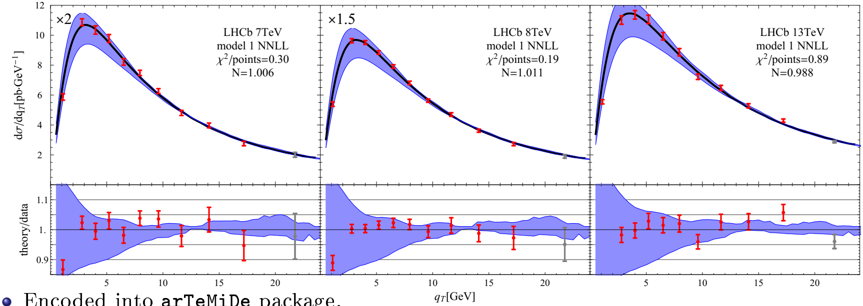
NNLL



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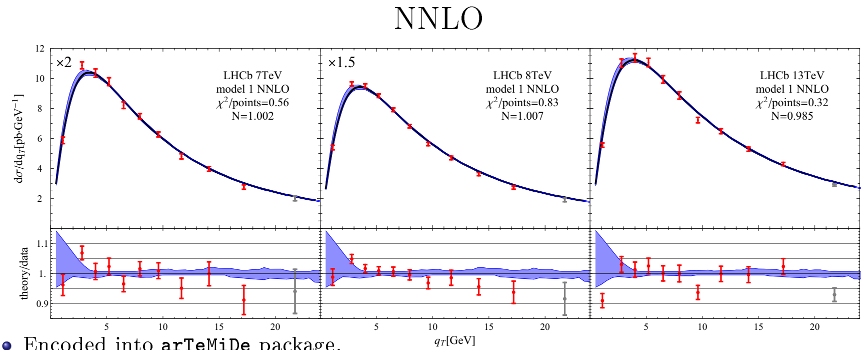

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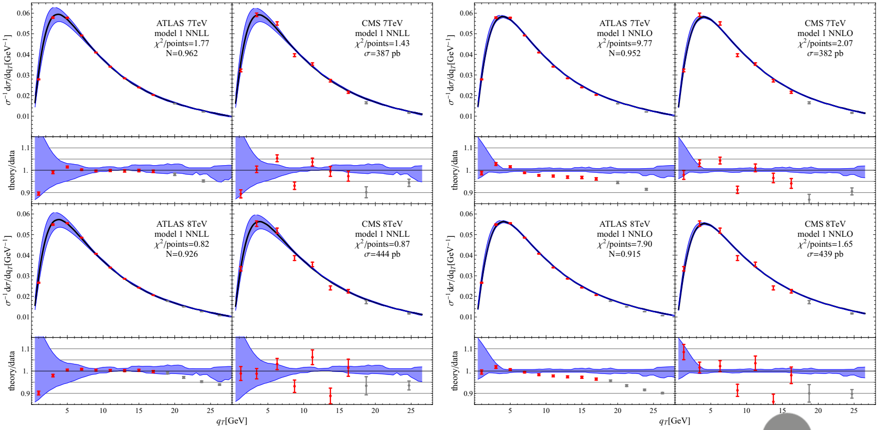
Experiment (ATLAS) has better error-bars then the theory.

NNLL($\sim$ `ResBos`,`DYres`) $\rightarrow$ NNLO($\sim$ `arTeMiDe`)



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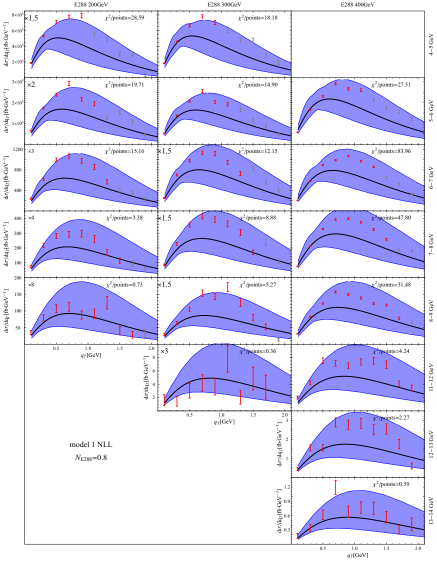
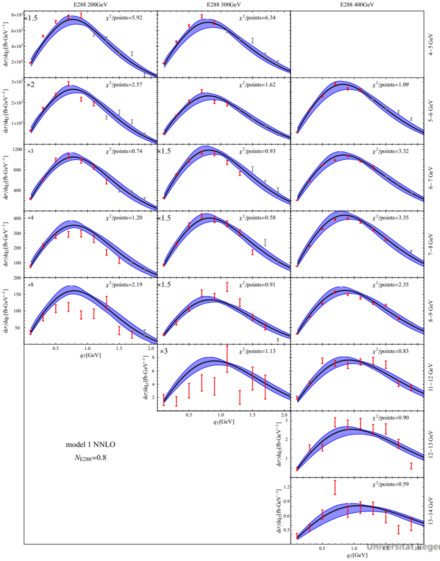
NLL → NNLO



E288 200GeV
E288 300GeV
E288 400GeV
χ²/points=28.59
χ²/points=18.18
χ²/points=19.71
χ²/points=14.90
χ²/points=27.51
χ²/points=15.16
χ²/points=12.15
χ²/points=83.96
χ²/points=3.38
χ²/points=8.88
χ²/points=47.80
χ²/points=0.73
χ²/points=5.27
χ²/points=31.48
χ²/points=0.36
χ²/points=4.24
χ²/points=2.27
χ²/points=0.59
4–5 GeV
5–6 GeV
6–7 GeV
7–8 GeV
8–9 GeV
11–12 GeV
12–13 GeV
13–14 GeV
dσ/dq_T[fb·GeV⁻¹]
q_T[GeV]
model 1 NLL
N_E288=0.8



E288 200GeV
E288 300GeV
E288 400GeV
χ²/points=5.92
χ²/points=6.34
χ²/points=2.57
χ²/points=1.62
χ²/points=1.09
χ²/points=0.74
χ²/points=0.93
χ²/points=3.32
χ²/points=1.20
χ²/points=0.58
χ²/points=3.35
χ²/points=2.19
χ²/points=0.91
χ²/points=2.35
χ²/points=1.13
χ²/points=0.83
χ²/points=0.90
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q_T[GeV]
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N_E288=0.8

However, rAD is not a usual UV anomalous dimension it has NP part

$$\mathcal{D} = \underbrace{\mathcal{D}^{[0]}(\ln(\mathbf{b}^2))}_{\text{pertrubative}} + g_K \mathbf{b}^2 + ...$$

- g_K (in some notation g_2) is a fundamental parameter for rapidity evolution. Universal for all TMD processes. But dependents on "prescription"

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Large-g_K

ResBos
[Nadolsky,etc;02-today]
based on b^*-prescription

$$g_K \sim 0.17 - 0.32\text{GeV}^2$$

DYRes,DYqT
[Catani,Grazzini,etc;08-today]
based on b^*-prescription/a_s-prescription

$$g_K \sim 0.3(?)\text{GeV}^2$$

Joined fit of TMDFF+TMDPDF
[Bacchetta,etc, 170310157]
based on b^{**}-prescription

$$g_K \sim 0.13\text{GeV}^2$$

Small-g_K

[D'Alesio,Scimemi,etc,1407.3311]
based on "resummation"

$$g_K = 0.0 - 0.03\text{GeV}^2$$

arTeMiDe
[I.Scimemi,AV,appear soon]
based on ζ-prescription

$$g_K = 0.0 - 0.015\text{GeV}^2$$

The only theoretical estimation
large-β_0 calculation
[I.Scimemi,AV,1609.06047]

$$g_K = (0.01 \pm 0.03)\text{GeV}^2$$

Prescriptions

- b^*-prescription, which is the most popular, introduces "uncontrollably" power corrections.

$$\mathbf{b} \to \frac{\mathbf{b}}{\sqrt{1+\frac{\mathbf{b}^2}{b_{\max}^2}}}$$

- ζ-prescription (used in `arTeMiDe`) matches the UV evolution and rapidity evolutions such that at large-b they cancel each other (no artificial corrections).
- At the moment, very precise (in the sense of theoretical errors) fit could not distinguish the non-zero g_K from the correction in OPE (although they have very different Q- and x-dependence.

$$F \sim (Q^2 b^2)^{-\mathcal{D}-g_K \mathbf{b}^2}(C(x)+\lambda_2 \mathbf{b}^2 C_2(x)) \otimes q(x)$$

$$\frac{\chi^2}{309\text{points}} = 1.86$$

$$g_K = 0.015 \pm 0.002\text{GeV}^2$$

$$\lambda_2 = 0$$

$$\frac{\chi^2}{309\text{points}} = 1.77$$

$$g_K = 0.0$$

$$\lambda_2 = -0.038 \pm 0.003\text{GeV}^2$$

$$\frac{\chi^2}{309\text{points}} = 1.75$$

$$g_K = 0.006 \pm 0.0005\text{GeV}^2 \qquad \lambda_2 = -0.0269 \pm 0.008\text{GeV}^2$$

Conclusion

- The rapidity divergences are alike UV divergences, and can be renormalized (in SCET II). It completes the proof of the TMD factorization theorem, and factorization for DPD, and multiPDs.
- Soft/Rapidity anomalous dimension correspondence

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- It has been checked up to three-loop order (!) for N=2 case (TMD), and up to two-loop order in the general case.
- It has multiple theoretical consequences.
- In particular, (I think) it make us hope for a realistic lattice formulation of TMDs.
- Current phenomenological state does not distinguish the g_K from the power correction in small-b OPE.

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