Structure of rapidity divergences in soft factors & rapidity renormalization theorem

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based on [1707.07606]

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General structure of the factorization theorems

The modern factorization theorems have the following general structure

$$\underbrace{\frac{d\sigma}{dX}}_{cross-X} = \underbrace{H}_{\substack{\text{Hard part}\\ \text{perturbative}}} \times \underbrace{f_1 \otimes ... \otimes J_2}_{\substack{\text{Parton distributions}\\ \text{jet-functions, etc}\\ \text{Non-pertrubative}\\ \text{universal}}} \times \underbrace{S}_{\substack{\text{Soft factor(s)}\\ \text{perturbative ?}}} + \substack{\text{Some power}\\ \text{suppressed terms}}$$

- This is a typical result of field mode separation
- Individual terms in the product are singular, and requires redefinition/refactorization
- The next factorization step is to factorize the divergences in soft factors
- The SF factorization is essential for low*er*-energy studies (e.g. SIDIS) and for resummation

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Outline of talk

- Example 1: Soft factor and rapidity decomposition for TMD factorization
- Example 2: Soft factor and rapidity decomposition for Double-Drell-Yan (and multi DY) ([AV,1608.04920])
- Structure of singularities in soft factors
- Proof of renormalization of rapidity divergences using *conformal transformation*
- Some consequences: factorization for mutiDY, correspondence between SAD and RAD, etc.

Example I: TMD factorization

TMD factorization

TMD factorization ($Q^2 \gg q_T^2$) gives us the following expression

$$\frac{d\sigma}{dQ dy\, d^2q_T} \sim \int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

TMD factorization

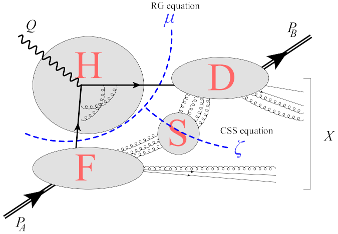
$$\frac{d\sigma}{dQ dy\, d^2q_T} \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\;\; \Phi_{h1}(z_1,b_T) S(b_T) \Delta_{h_2}(z_2,b_T) + Y$$

TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

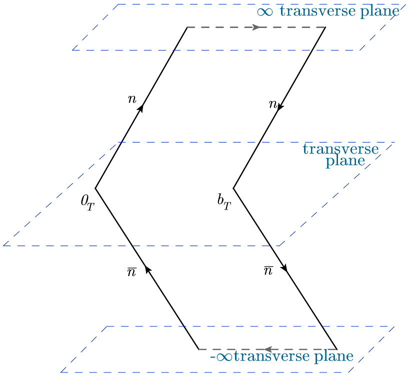
TMD PDF (singular)



All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n\cdot\bar{n}) = 1$$

Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Multiple divergences!

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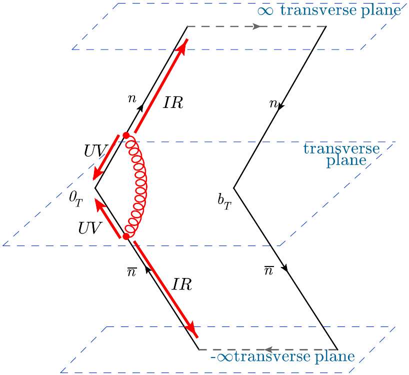
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$

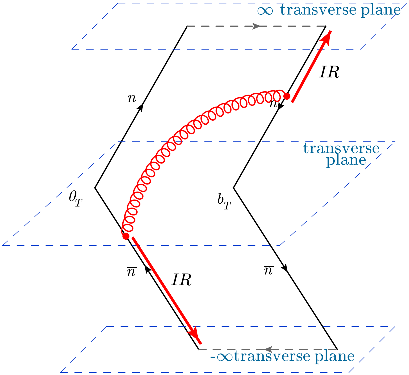


$$\begin{aligned}\int \quad dxdy \quad & D(x-y)\\ =\quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-}\\ =\quad & \int_0^\infty \frac{dx^+}{x^+}\int_0^\infty \frac{dy^-}{y^-}\\ =\quad & (\mathrm{UV}+\mathrm{IR})\,(\mathrm{UV}+\mathrm{IR})\end{aligned}$$

Some people set it to zero.

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned} \int & dxdy \quad D(x-y) \\ &= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)} \\ &= \text{IR at } x, y \to \infty \end{aligned}$$

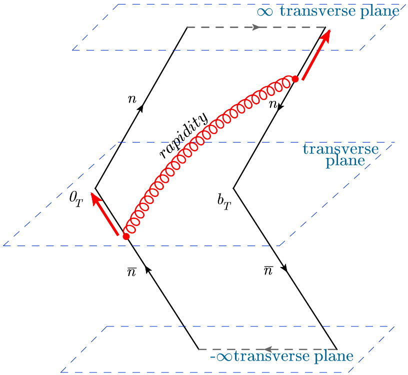
However, it exactly cancels IR from the previous diagram

Proved at all orders,

e.g.[Echevarria,Scimemi,AV,1511.05590]

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \quad \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \quad \text{rap. div. at } \lim_{\lambda\to 0}\{x=\lambda, y=\lambda^{-1}\}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

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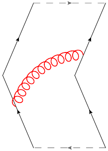
δ-regularization + dimension regularization($\epsilon > 0$)

$$P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma n^\mu A_\mu(n\sigma)e^{-\delta\sigma}\right)$$

Nice, and continent composition of regularizations, that clear separate divergences.



$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{e^{-\delta^+ y^-} e^{-\delta^- x^+}}{(2x^+y^- + \mathbf{b}_T^2)^{1-\epsilon}}$$

$x^+ \to zL, \quad y^- \to L/z$

In this calculation scheme every divergece takes particular form

$$\left(\frac{\mathbf{b}^2}{4}\right)^\epsilon \left(\ln\left(\delta^+\delta^- \frac{\mathbf{b}^2 e^{2\gamma_E}}{4}\right) - \psi(-\epsilon) - \gamma_E\right) + \left(\delta^+\delta^-\right)^{-\epsilon}\Gamma^2(-\epsilon)$$

δ-regularization + dimension regularization$(\epsilon > 0)$

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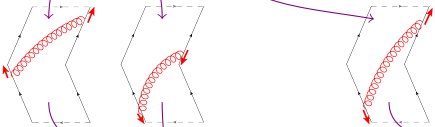
Nice, and continent composition of regularizations, that clear separate divergences.

$x^+ \to zL, \quad y^- \to L/z$

$$\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{2LdL}{(L^2+\mathbf{b}^2)^{1-\epsilon}} e^{-L(z\delta^+ + \delta^-/z)}$$



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Typical expression

Generally (say at NNLO) one expects the following form (finite ϵ, $\delta \to 0$)

$$S^{[2]} = \underbrace{\overbrace{A_1\boldsymbol{\delta}^{-2\epsilon} + A_2\boldsymbol{\delta}^{-\epsilon}\boldsymbol{B}^{\epsilon}}^{\text{IR}} + \boldsymbol{B}^{2\epsilon}\Big(A_3\ln^2(\boldsymbol{\delta}B)}_{\text{cancel in sum of diagram}} + A_4\ln(\boldsymbol{\delta}B) + A_5\Big)$$

- Terms $\sim (\boldsymbol{\delta})^{-\epsilon}$ cancel exactly at all orders (proved!) see e.g.[AV;1707.07606,app.A]
- A_3 cancels due to Ward identity (alike leading UV pole for cusp)(what about NNNLO?)

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

It allows to split rapidity divergences and define individual TMDs.

<u>Important note:</u> the structure is independent on ϵ

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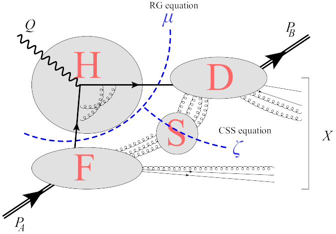
$$\exp(A\ln(\delta^+\delta^-)+B)=\exp\left(\frac{A}{2}\ln((\delta^+)^2\zeta)+\frac{B}{2}\right)\exp\left(\frac{A}{2}\ln((\delta^-)^2\zeta^{-1})+\frac{B}{2}\right)$$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\ \ \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities

$$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$$

$$d\sigma\sim\int d^2b_T\, e^{-i(qb)_T}H(Q^2)\ F(z_1,b_T;\zeta^+)\ D(z_2,b_T;\zeta^-)+Y$$



TMD PDF	TMD FF
$\sqrt{S}\Phi_{h_1}$	$\sqrt{S}\Delta_{h_2}$
(regular)	(regular)

The extra "factorization" introduces extra scale ζ.

And corresponded evolution equation

$$\zeta\frac{d}{d\zeta}F=\frac{A}{2}F=-\mathcal{D}F$$

Rapidity anomalous dimension (RAD)

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TMD evolution

TMD evolution is evolution in two-dimension plane

$$\begin{pmatrix} \frac{d}{d\ln\mu^2} \\ \frac{d}{d\ln\zeta} \end{pmatrix} F(x,b) = \begin{pmatrix} \gamma_F \\ -\mathcal{D} \end{pmatrix} F(x,b) \qquad = \begin{pmatrix} \text{UV part} \\ \text{rap.part} \end{pmatrix} F(x,b)$$

Nowadays (at NNLO) the largest error comes from the evolution [I.Scimemi,AV,1706.01473]

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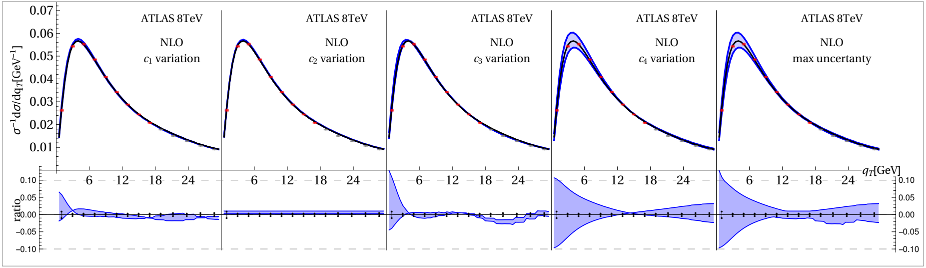
High-energy example: ATLAS 8 TeV (best precision)



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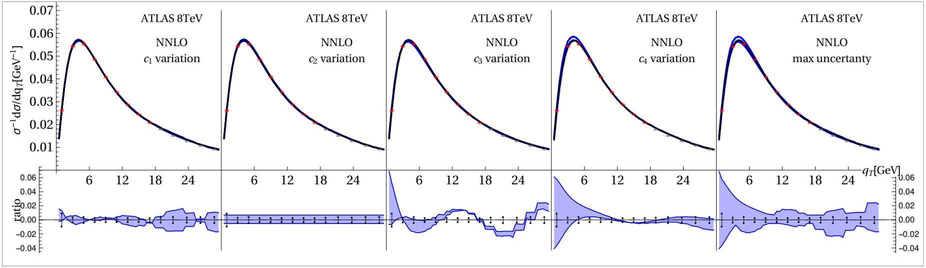

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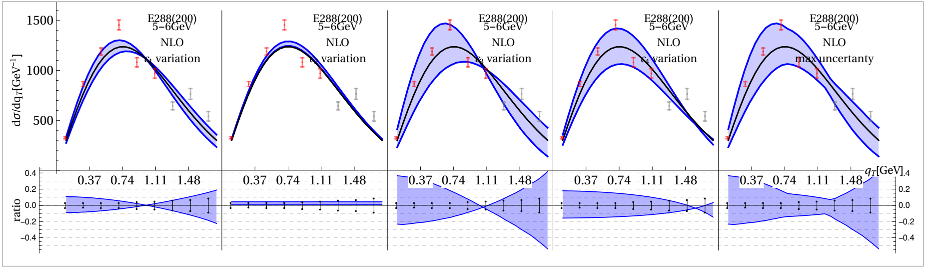

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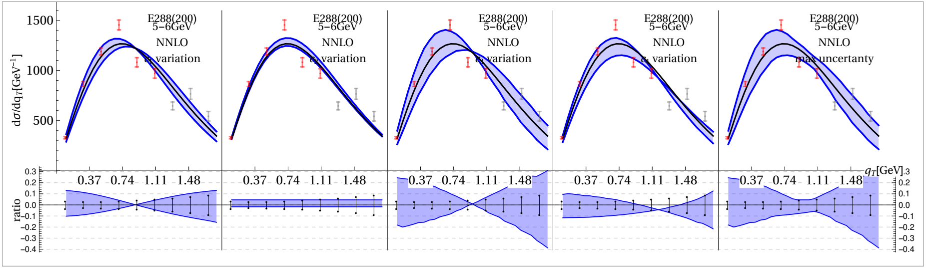

There is no proof of rapidity divergence factorization

- Checked at NNLO (two-loops) [Echevarria,Scimemi,AV,1511.05590],[Lübbert,Oredsson,Stahlhofen,1602.01829], [Li,Neill,Zhu,1604.00392]
- Nothing is known about large (finite) b (however, leading renormalon part also factorizes [Scimemi,AV,1609.06047]).

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Example II: Double-Drell-Yan scattering

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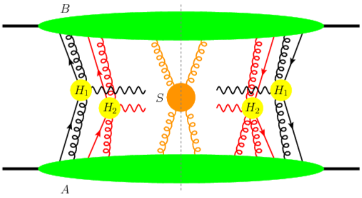


pictures from [1510.08696]

Double Drell-Yan scattering

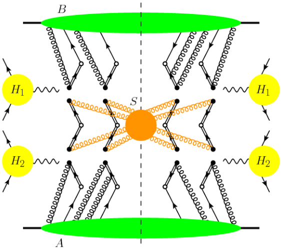
- Experimental status is doubtful
- Collinear part of factorization is proved [Diehl,et al,1510.08696]
- In many aspects similar to TMD factorization
- The same problem of rapidity factorization, but enchanted by matrix structure

DPD soft factor (very singular)

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\ F^A_{h1}(z_{1,2}, b_{1,2,3,4}) S^{AB}(b_{1,2,3,4}) \bar{F}^B_{h2}(z_{1,2}, b_{1,2,3,4}) + Y$$

power suppressed terms

DPD (singular)



Structure is similar to TMD Drell-Yan but now it contains COLOR

The soft factor is a matrix

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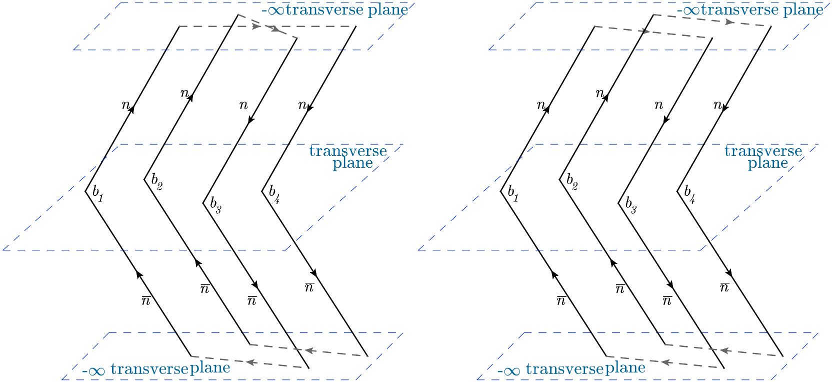
Color structure makes a lot of difference

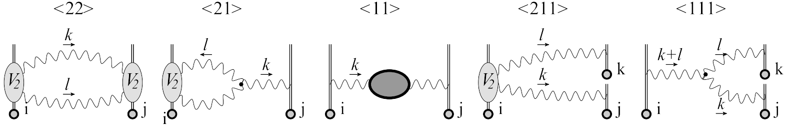
$$F^A_{h1} S^{AB} \bar{F}^B_{h2} \xrightarrow{\text{singlets}} (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix}$$

- Soft-factors $S^{\mathbf{ij}}$ are sum of Wilson loops and double Wilson loops (all possible connections).
- Soft-factors are non-zero even in the integrated case.



Evaluation at NNLO [AV,1608.04920]

- Brute force evaluation would lead a lot of (similar) diagrams.
- The better way is to compute the generating function [AV, 1406.6253, 1501.03316]



- All non-trivial three-Wilson line interactions cancel!
- The final result expresses via TMD soft factor only!

$$\text{TMD SF} \quad : \quad \ln S^{\text{TMD}} = \sigma(\mathbf{b})$$

$$\text{Single loop SF} \quad : \quad \ln S^{[4]} = \sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})+\sigma(\mathbf{b}_{14})+\sigma(\mathbf{b}_{23})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}) + \frac{C_A}{4C_F}(\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{14})-\sigma(\mathbf{b}_{23})+\sigma(\mathbf{b}_{24}))(\sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}))$$

$$\text{Double loop SF} \quad : \quad \ln S^{[1]} = \sigma(\mathbf{b}_{14})+\sigma(\mathbf{b}_{23})+\frac{1}{2}\left(\frac{C_A}{4C_F}-1\right)(\sigma(\mathbf{b}_{12})-\sigma(\mathbf{b}_{13})-\sigma(\mathbf{b}_{24})+\sigma(\mathbf{b}_{34}))^2$$

- This structure is independent on regularization procedure!

REMINDER: TMD factorization

$$S^{\mathrm{TMD}} = e^{\sigma(\mathbf{b})} = e^{\sigma^{+}(\mathbf{b})} e^{\sigma^{-}(\mathbf{b})}, \qquad \sigma^{\pm} = \frac{A}{2}\ln((\delta^{+})^2\zeta^{\pm 1}) + \frac{B}{2}$$

Matrix factorization of rapidity divergences

Using the decomposition above, inserting it into DPD SF we obtain matrix relation

$$S^{\mathrm{DPD}} = s^T(\ln(\delta^{+})) \cdot s(\ln(\delta^{-}))$$

$$s = \exp\left[\begin{pmatrix} A^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ A^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & A^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix} \ln(\delta) + \begin{pmatrix} B^{\mathbf{11}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{18}}(\mathbf{b}_{1,2,3,4}) \\ B^{\mathbf{81}}(\mathbf{b}_{1,2,3,4}) & B^{\mathbf{88}}(\mathbf{b}_{1,2,3,4}) \end{pmatrix}\right]$$

$A^{\mathbf{ij}}$ and $B^{\mathbf{ij}}$ are rather complicated non-linear compositions of TMD's A and B

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Finalizing DPD factorization

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\ (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix} + Y$$

spliting rapidity singularities

$S(b_{1,2,3,4}) \to s^T(b_{1,2,3,4};\zeta^+) s(b_{1,2,3,4};\zeta^-)$

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}]\, e^{-i(qb)_T} H_1(Q_1^2) H_2(Q_2^2)\ F(z_{1,2}, b_{1,2,3,4};\zeta^+) \bar{F}(z_{1,2}, b_{1,2,3,4};\zeta^-) + Y$$

DPD $(sF_{h_1})^T$ (regular)

DPD sF_{h_2} (regular)

Matrix rapidity evolution

$$\frac{dF(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)}{d\ln\zeta} = -F(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)\mathbf{D}(\mathbf{b}_{1,2,3,4},\mu)$$

where $\mathbf{D}$ is matrix build (linearly) of TMD rapidity anomalous dimensions. E.g.

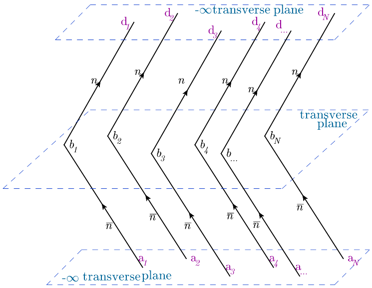
$$D^{\mathbf{18}} = \frac{\mathcal{D}(\mathbf{b}_{12}) - \mathcal{D}(\mathbf{b}_{13}) - \mathcal{D}(\mathbf{b}_{24}) + \mathcal{D}(\mathbf{b}_{34})}{\sqrt{N_c^2 - 1}}$$

Let's look at multi-parton scattering

- Just as double-parton, but multi..(four WL's $\to$ arbitrary number WL's)
- Too many color-singlets, better to work with explicit color indices (color-multi-matrix)

$$\Sigma^{(a_1 \dots a_N);(d_1 \dots d_N)}(\mathbf{b}_1, ..., \mathbf{b}_N) = \mathbf{\Sigma}(\mathbf{b}_{1,..,N})$$

$$\mathbf{\Sigma}(\{b\}) = \langle 0|T\{[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_N)\dots[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_1)\}|0\rangle$$



Color-matrix notation

- All color flow in the same direction
- i'th WL has generator $\mathbf{T}_i$
- In total the soft factor is color-neutral

$$\sum_i \mathbf{T}_i = 0.$$

- Color-neutrality $\to$ gauge invariance + cancellation IR singularities

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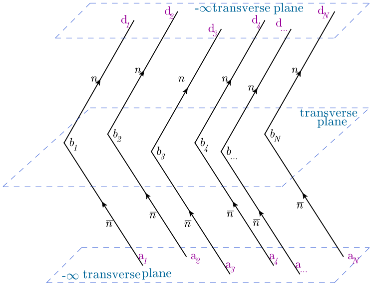

Result at NNLO is amazingly simple

$$\mathbf{\Sigma}(\mathbf{b}_{1,..,N}) = \exp\left(-\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A\sigma(\mathbf{b}_{ij}) + \mathcal{O}(a_s^3)\right)$$

- $\mathbf{T}_i^A\mathbf{T}_j^A$ = "dipole"
- $\mathcal{O}(a_s^3)$ contains also "color-multipole" terms
- Rapidity factorization for dipole part is straightforward (assuming TMD factorization)

The factorization of rapidity divergences seems to exists.

- True?
- In which case it is violated?
- Criterion of rapidity factorization?
- Can we have some other benefit from it?
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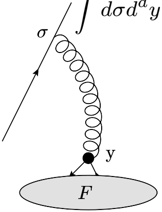
Step I: Origin and counting rules for rapidity divergences.

Consider a diagram with gluon radiated to/by light-like WL.

$$\int d\sigma d^d y \frac{F(y)}{[-(\sigma n - y)^2 + i0]^p}$$



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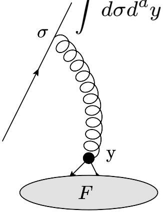
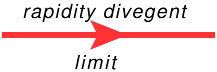
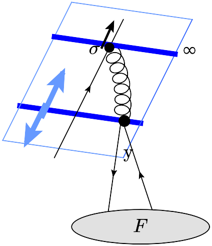






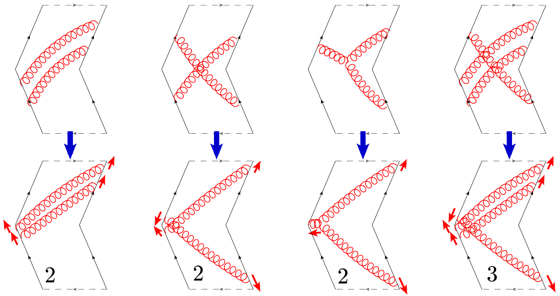
$$\int^{\infty} \frac{d\sigma}{\sigma} \int d^d y \frac{F(y^+ = \text{const})}{[-(2y^+ + y_\perp^2)^2 + i0]^p}$$

- Divergent configuration is then gluon radiated to the far-end of WL, from the "plane" transverse to the direction of WL.
- The position of the plane is unimportant.
- Let us send the "plane" close to infinity and associate rapidity divergences with this plane.

Counting rules for rapidity divergences

1. Consider a subgraph with attached to WL gluon.
2. Send attached end to infinity, another end to the transverse (to WL!) plane.
3. If it is possible, +1 power of rapidity divergence, remove this subgraph. If not consider a larger subgraph.

Examples of high degree divergences



$\ln^2 \delta^+\delta^-$ $\quad$ $\ln^2 \delta^+\delta^-$ $\quad$ $\ln^2 \delta^+\delta^-$ $\quad$ $\ln^3 \delta^+\delta^-$

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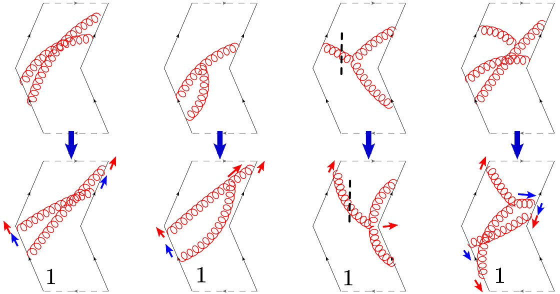
Examples of low degree divergences



$\ln \delta^+ \delta^-$ $\quad \ln \delta^+ \delta^-$ $\quad \ln \delta^+ \delta^-$ $\quad \ln \delta^+ \delta^-$

Important lessons

- The rapidity divergences are associated with planes! Not with directions.
- The planes defined unambiguously. However, in (TMD-like) soft factor we have two light-like directions, i.e. two planes and two independent types of rapidity divergences. The requirement of non-intersection of these planes defines them unambiguously.
- Counting rules are just alike UV divergence counting rules.

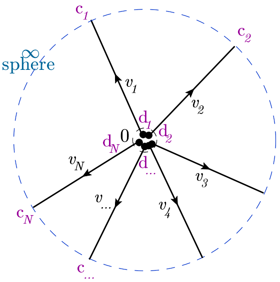
Step II: From planes to points

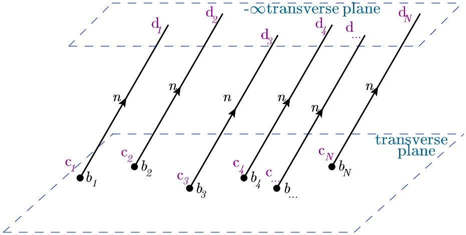
Rapidity divergences associated with transverse planes (or better to say with the layer between the transverse plane and infinity). Let us construct a conformal transformation which maps it to the point (to a sphere).

Conformal-stereographic transformation

$$\mathcal{C}_{\bar{n}} : \{x^+, x^-, x_\perp\} \to \{\frac{-1}{2a}\frac{1}{\lambda + 2ax^+}, x^- + \frac{ax_\perp^2}{\lambda + 2ax^+}, \frac{x_\perp}{\lambda + 2ax^+}\}$$

- Translation – special conformal transformation (along n) – Translation
- a and λ are free parameters





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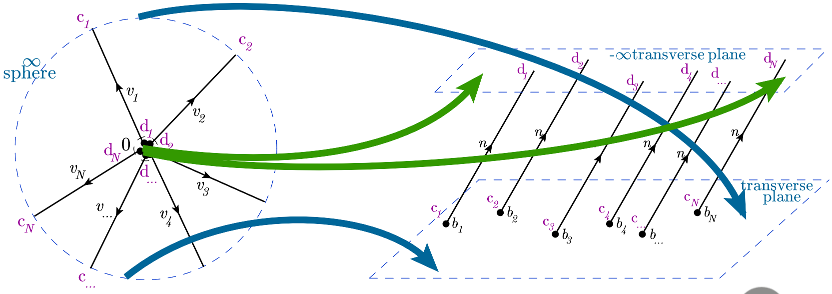

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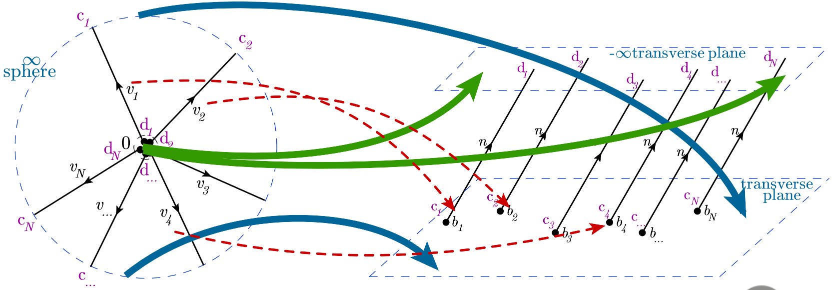

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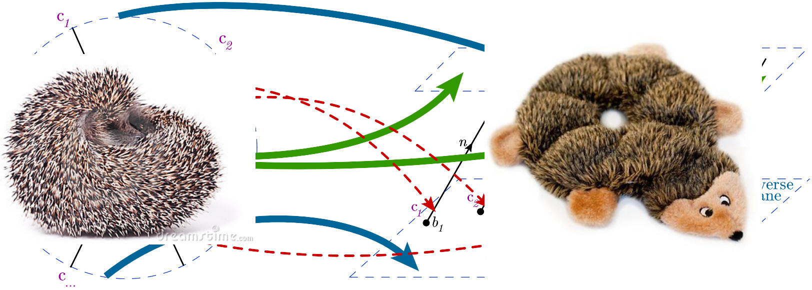


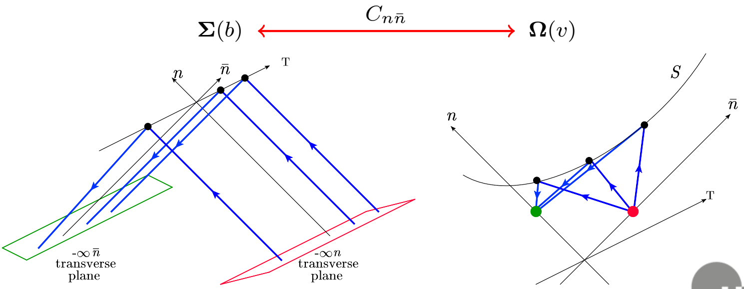
Composition of two conformal-stereographic transformations

$$C_{n\bar{n}} = \mathcal{C}_n \mathcal{C}_{\bar{n}} = \mathcal{C}_{\bar{n}} \mathcal{C}_n$$

With the special choice of parameters

$$a\lambda < 0, \qquad \bar{a}\bar{\lambda} < 0, \qquad (a\bar{a})^2 < \frac{1}{2\rho_T^2} \qquad \rho_T^2 > \max\{b_i^2\},$$

any DY-like soft factor transforms to a compact object.



In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization

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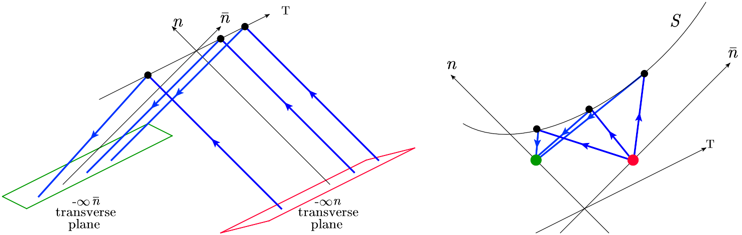
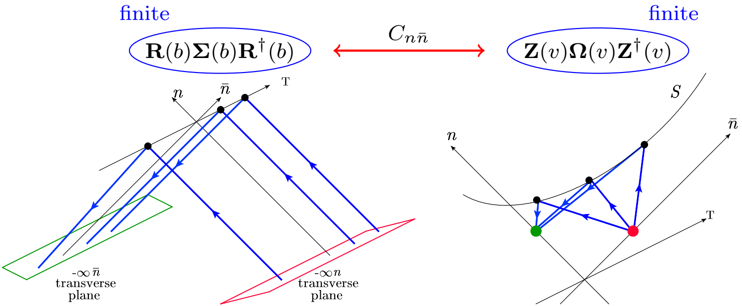
- The UV renormalization imposes rapidity divergence renormalization

$$\mathbf{R}(b)\mathbf{\Sigma}(b)\mathbf{R}^\dagger(b) \xleftrightarrow{\;C_{n\bar{n}}\;} \mathbf{Z}(v)\mathbf{\Omega}(v)\mathbf{Z}^\dagger(v)$$



In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization
- There are also UV renormalization factors in cusps (we omit them for a moment)



RDRT in conformal theory

In a conformal theory rapidity divergences can be removed (*renormalized*) by a multiplicative factor.

$$C_{n\bar{n}}^{-1}\left(\mathbf{Z}(\{v\},\mu)\right) = \mathbf{R}_n(\{b\},\nu^+)$$

Rapidity anomalous dimension (RAD)

$$\mathbf{D}(\{b\}) = \frac{1}{2}\mathbf{R}_n^{-1}(\{b\},\nu^+)\nu^+\frac{d}{d\nu^+}\mathbf{R}_n(\{b\},\nu^+),$$

In CSS notation it is $-K$, in [Becher,Neubert] $F_{q\bar{q}}$, in SCET literature γ_ν.

(In CFT) DY-like Soft factors expresses as

$$\mathbf{\Sigma}(\{b\},\delta^+,\delta^-) \quad = \quad e^{2\mathbf{D}(\{b\})\ln(\delta^+/\nu^+)}\overbrace{\mathbf{\Sigma}_0(\{b\},\nu^2)}^{\text{finite}}e^{2\mathbf{D}^\dagger(\{b\})\ln(\delta^-/\nu^-)},$$

Step III: From conformal theory to QCD

QCD at the critical point

QCD is conformal in $4-2\epsilon^*$ dimensions

$$\beta(\epsilon^*)=0, \qquad \Rightarrow \qquad \epsilon^* = -a_s\beta_0 - a_s^2\beta_1 - ....$$

It is very useful trick, allows to restore "conformal-violating" terms, see e.g.[Braun,Manashov,1306.5644]

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Thus, at $4-2\epsilon^*$ dimensions, the rapidity renormalization theorem works.

RTRD works at any finite order of QCD

Proof by induction

- Important input: Counting of rap.div. is independent on number of dimensions
- Important input: At 1-loop QCD is conformal = RTRD hold.
- (1) All Leading divergences cancel by R.
- (2) Make shift $\epsilon^* \to \epsilon^* + \beta_0 a_s$.
- (3) Modify R such that next-to-leading divegences cancel (it can be done perturbatively, thanks to a_s)
- Repeat (2-3) N times, and got renormalization at a_s^{N+1} order.

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Soft factor has the form

$$\mathbf{\Sigma}(\{b\},\delta^+,\delta^-) \quad = \quad e^{2\mathbf{D}(\{b\})\ln(\delta^+/\nu^+)}\overbrace{\mathbf{\Sigma}_0(\{b\},\nu^2)}^{\text{finite}}e^{2\mathbf{D}^\dagger(\{b\})\ln(\delta^-/\nu^-)},$$

$$\mathbf{D}_{\text{QCD}} \neq \mathbf{D}_{\text{CFT}}$$

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Example then it does not work (no factorization?)

There are talks about "dipole-like" distributions that could appear in processes like $pp \to hX$

e.g. [Boer,et al,1607.01654]

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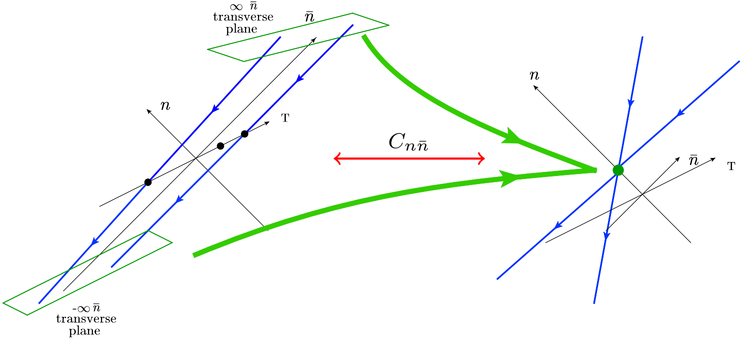

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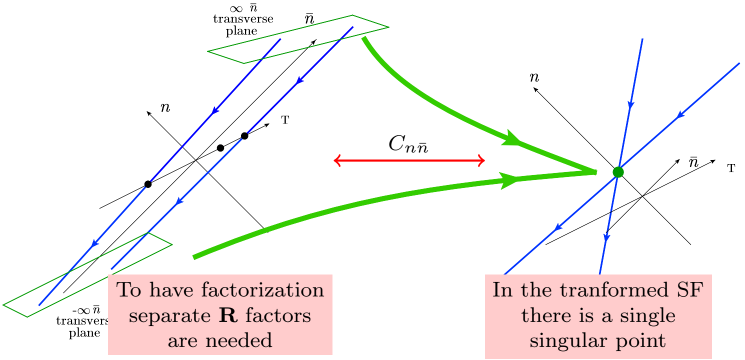


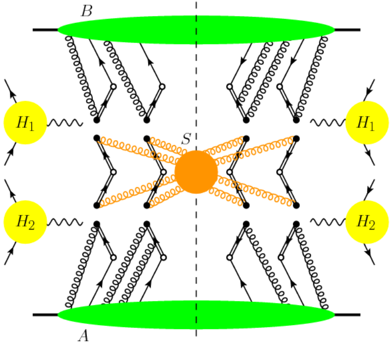
- The renormalization of dipole recouple colors → extra gauge link → ala BK equation.

Consequences

- Factorization for multi-Drell-Yan process
- Generalized CSS equation
- Correspondence between soft and rapidity anomalous dimensions
- Constraints of soft anomalous dimension.
- Universality of DY and SIDIS TMD soft factors
- Many others ...

Factorization of multi-parton scattering

$$\frac{d\sigma}{dX} \sim \prod_i H_i^{f\bar{f}}\left(\frac{Q_i}{\mu}\right)\int db_i\, e^{-i(q_i b_i)}\ \tilde{\bar{F}}_{\bar{f}}(\{\bar{x}\},\{b\},\mu)\boldsymbol{\Sigma}(\{b\})\tilde{F}_f(\{x\},\{b\},\mu)+Y$$



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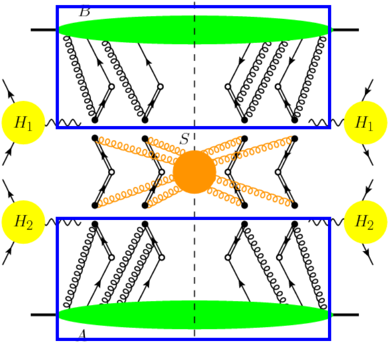

$$\tilde{F}^{\{a\}}(\{x\},\{b\}) = \int \{d^3 y_i\}\langle h|\xi_1^{a_1}(y_1)...\xi_N^{a_N}(y_N)|h\rangle$$

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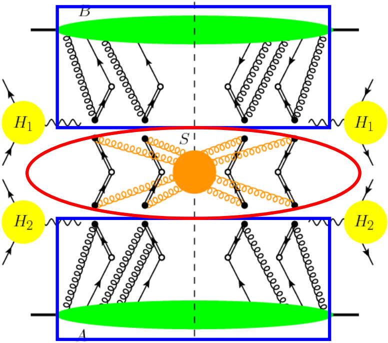

$$\boldsymbol{\Sigma}(\{b\}) = \langle 0|(\boldsymbol{\Phi}_{-n}\boldsymbol{\Phi}_{-\bar n})(b_1)\ldots(\boldsymbol{\Phi}_{-n}\boldsymbol{\Phi}_{-\bar n})(b_N)|0\rangle$$

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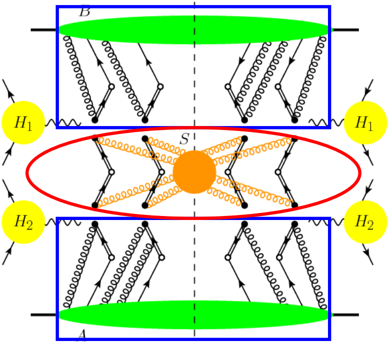

$$\boldsymbol{\Sigma}(\{b\}) = \mathbf{R}_n^{-1}(\{b\},\nu^+)\boldsymbol{\Sigma}_0(\nu^2)\mathbf{R}_{\bar{n}}^{\dagger-1}(\{b\},\nu^-)$$

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Factorization of multi-parton scattering

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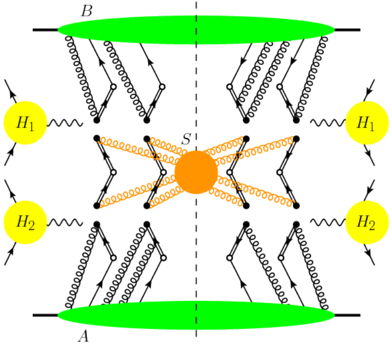

$$F_f(\{x\},\{b\},\nu^+) =$$
$$\boldsymbol{\Sigma}_0(\{b\},\nu^2)\mathbf{R}^{\dagger-1}(\{b\},\nu^-)\tilde{F}_f(\{x\},\{b\})$$

Finite multiPD

Finite multiPD

- The definition of collinear matrix element include zero-bin subtraction

$$\tilde{F}(\{x\},\{b\},\mu,\delta^-) = \overbrace{\mathbf{\Sigma}^{-1}(\mu;\delta^+,\delta^-)}^{\text{z.b.}} \times \overbrace{\tilde{F}^{\text{us}}(\{x\},\{b\},\mu,\delta^+)}^{\text{unsubtracted}}$$

- Zero-bin partially cancel rapidity renormalization

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Rapidity evolution

The rapidity evolution is matrix evolution

$$\nu^+\frac{d}{d\nu^+}F(\{x\},\{b\},\mu,\nu^+) = \frac{1}{2}\mathbf{D}(\{b\},\mu) \times F(\{x\},\{b\},\mu,\nu^+).$$

Boost invariant variables

- We define boost invariant variables

$$\zeta = 2(p^+)^2\frac{\nu^-}{\nu^+}, \qquad \bar{\zeta} = 2(p^-)^2\frac{\nu^+}{\nu^-}, \qquad \zeta\bar{\zeta} = (2p^+p^-)^2$$

- In terms of these variables

$$\zeta\frac{d}{d\zeta}F_{\{f\}\leftarrow h}(\{x\},\{b\},\mu,\zeta,\nu^2) = -\mathbf{D}^{\{f\}}(\{b\},\mu)\times F_{\{f\}\leftarrow h}(\{x\},\{b\},\mu,\zeta,\nu^2). \qquad (1)$$

  ν^2 is some IR scale.

- Generalized CS equation

$$\mu^2\frac{d}{d\mu^2}\mathbf{D}(\{b\},\mu) = \frac{1}{4}\sum_{i=1}^{N}\Gamma^i_{\text{cusp}}\mathbf{I}.$$

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Scheme dependence

We can add arbitrary finite terms to the rapidity redefinition

$$F(\{x\},\{b\},\zeta,\nu^2) \to \mathbf{S} \times F(\{x\},\{b\},\zeta,\nu^2).$$

It is equivalent to the scheme dependence for UV renormalization.

Natural definition

The soft factor remnant $\mathbf{\Sigma}_0$ can be absorbed to the definition of multiPD:

$$\mathbf{S}(b,\mu,\nu^2)\mathbf{\Sigma}_0(\{b\},\mu,\nu^2)\mathbf{S}^T(b,\mu,\nu^2) = \mathbf{I}.$$

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In the TMD case it leads to the standard definition

$$F(x,b,\mu,\zeta) \quad = \quad \sqrt{\Sigma_{\text{TMD}}\left(b, \frac{\delta^+}{\sqrt{2}p^+}\sqrt{\zeta}, \frac{\delta^+}{\sqrt{2}p^+}\sqrt{\zeta}\right)}\tilde{F}(x,b,\delta^+).$$

Soft/rapidity anomalous dimension correspondence

The equivalence (under conformal transformation) between **Z** and **R** implies the equality between corresponding anomalous dimensions

$$\gamma_s(\{v\}) = 2\mathbf{D}(\{b\})$$

It has been observed in [Li,Zhu,1604.01404].

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

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In QCD

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- Exact relation!
- Connects different regimes of QCD

→ Lets test it.

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$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

How to use it?

- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$
- $\epsilon^* = 0 - a_s\beta_0 - a_s^2\beta_1 - a_s^3\beta_2 - ....$
- We can compare order by order in PT

$$\begin{aligned}
\mathbf{D}_1(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_1(\{v\}), \\
\mathbf{D}_2(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_2(\{v\}) + \beta_0\mathbf{D}_1'(\{b\}), \\
\mathbf{D}_3(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_3(\{v\}) + \beta_0\mathbf{D}_2'(\{b\}) + \beta_1\mathbf{D}_1'(\{b\}) - \frac{\beta_0^2}{2}\mathbf{D}_1''(\{b\}),
\end{aligned}$$

TMD rapidity anomalous dimension

- $N = 2$, no matrix structure, $\boldsymbol{B} = \frac{b^2}{4}$, $L = \ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right) \leftrightarrow \ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)} = L + 0$

NLO | NLO

$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right) = \mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}$ $a_s\mathcal{D}^{(1)}$

$$\gamma_s(\{v\}) = 2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{\nu^2}\right)+0\right) = a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{4e^{-2\gamma_E}}\right)+0\right)$$

Obvious relation, QCD is conformal at leading order.

$\nu^2 = 4e^{-2\gamma_E}$

TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\boldsymbol{B}=\frac{b^2}{4}$, $L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow\ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+\ldots)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$

NLO | NLO

N^{2}LO

$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}$

$a_s\mathcal{D}^{(1)}+a_s^2?\mathcal{D}^{(2)}$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$..+a_s^2\left(\Gamma_2L+\gamma^{(2)}\right)=..+2a_s^2\left(\mathcal{D}^{(2)}-2\beta_0(L^2+\zeta_2)\right)$$

We found 2-loop rapidity anomalous dimension

$$\mathcal{D}^{(2)}_{L=0}=\left(\frac{404}{27}-14\zeta_3\right)C_A-\frac{112}{27}\frac{N_f}{2}$$

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TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\quad \boldsymbol{B}=\frac{b^2}{4},\ L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow \ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$ **NLO**

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+\dots)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$ **N²LO**

$\gamma_s^{(3)}=[\frac{245}{3}C_A^2+\dots]L+$
$+(-192\zeta_5 C_A^2+\dots+\frac{2080}{729}N_f^2)$ **N³LO**

NLO $-2\left(\boldsymbol{B}^{\epsilon}\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

N²LO $\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)=\mathcal{D}^{(2)}$
$+\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\left(\frac{3(4-3\epsilon)}{2\epsilon}C_A-N_f\right)\Big)$
$+\boldsymbol{B}^{\epsilon}\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0+\frac{\beta_0}{2\epsilon^2}-\frac{\Gamma_1}{2\epsilon}$

[Echevarria,Scimemi,AV,1511.05590]

$$\underbrace{a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}+a_s^3\gamma_s^{(3)}} \longrightarrow \gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*) \longleftarrow \underbrace{a_s\mathcal{D}^{(1)}+a_s^2\mathcal{D}^{(2)}+a_s^3?\mathcal{D}^{(3)}}$$

Expand in a_s

$$..+a_s^3\left(\Gamma_3 L+\gamma^{(3)}\right)=..+2a_s^2\Big[\mathcal{D}^{(3)}-\frac{2\beta_0^2}{3}L^3-\left(\frac{\beta_0\Gamma_1}{2}+\beta_1\right)L^2+\beta_0(\gamma_1-2\beta_0\zeta_2)L$$
$$-\beta_0\Gamma_1\frac{\zeta_2}{4}-\zeta_2\beta_1+\frac{2\beta_0^2}{3}\left(\zeta_3-\frac{82}{9}\right)+26\beta_0C_A\left(\zeta_4-\frac{8}{27}\right)\Big]$$

We found 3-loop rapidity anomalous dimension

$$\mathcal{D}^{(3)}_{L=0}=C_A^2\left(\frac{297029}{1458}+\frac{88}{3}\zeta_2\zeta_3+\dots+96\zeta_5\right)+\dots+C_FN_f\left(\frac{-152}{9}\zeta_3-8\zeta_4+\frac{11711}{54}\right)$$

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$$\mathcal{D}^{(3)}_{L=0} = -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3-\frac{88}{3}\zeta_2\zeta_3-192\zeta_5-\frac{297029}{729}+\frac{6392}{81}\zeta_2+\frac{154}{3}\zeta_4\right)$$
$$-\frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3+\frac{62626}{729}-\frac{824}{81}\zeta_2+\frac{20}{3}\zeta_4\right)-$$
$$\frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3+\frac{1711}{27}-16\zeta_4\right)-\frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3-\frac{1856}{729}\right)$$

- Coincides with the one calculated directly [Li,Zhu,1604.01404]
- The logarithmic structure of rapidity anomalous dimension also restored

$$\mu^2\frac{d}{d\mu^2}\mathcal{D}(a_s(\mu),\mathbf{b})=\frac{\Gamma_{cusp}(a_s(\mu))}{2}$$

vs.

$$\nu^2\frac{d}{d\nu^2}\gamma_s(\nu,v)=\frac{\Gamma_{cusp}}{2}$$

UV anomalous dimensions independent on ϵ. UV anomalous dimension of rapidity anomalous dimension also.

Quadrupole part of SAD

$$\boldsymbol{\gamma}_s(\{v\}) = -\frac{1}{2}\sum_{[i,j]}\mathbf{T}_i^A\mathbf{T}_j^A\gamma_{\text{dipole}}(v_i\cdot v_j) - \sum_{[i,j,k,l]} if^{ACE} if^{EBD}\mathbf{T}_i^A\mathbf{T}_j^B\mathbf{T}_k^C\mathbf{T}_l^D\mathcal{F}_{ijkl} - \sum_{[i,j,k]}\mathbf{T}_i^{\{AB\}}\mathbf{T}_j^C\mathbf{T}_k^D if^{ACE} if^{EBD} C + \mathcal{O}(a_s^4),$$

Quadrupole part has been calculated in [Almelid,Duhr,Gardi;1507.00047]

$$\tilde{C} = a_s^3\left(\zeta_2\zeta_3 + \frac{\zeta_5}{2}\right) + \mathcal{O}(a_s^4),$$

$$\tilde{\mathcal{F}}_{ijkl}(\{b\}) = 8a_s^3\mathcal{F}(\tilde{\rho}_{ikjl}, \tilde{\rho}_{iljk}) + \mathcal{O}(a_s^4),$$

Quadrupole part of RAD

- Color structures are not affected by ϵ^*
- Quadrupole contribution depends only on conformal ratios

$$\rho_{ijkl} = \frac{(v_i\cdot v_j)(v_k\cdot v_l)}{(v_i\cdot v_k)(v_j\cdot v_l)} \quad\leftrightarrow\quad \tilde{\rho}_{ijkl} = \frac{(b_i-b_j)^2(b_k-b_l)^2}{(b_i-b_k)^2(b_j-b_l)^2}$$

The correspondence between SAD and RAD can be used also to constraint the SAD. It seems that structure of RAD (diagrammatically) is simpler.

Color-structure of soft anomalous dimension

As a consequence of Lorentz invariance one has

$$\mathbf{\Sigma}(\{b\}) = \mathbf{\Sigma}^{\dagger}(\{b\})$$

It implies that RAD has only even color-multipoles

$$\mathbf{D}(\{b\}) \quad = \quad \sum_{\substack{n=2 \\ n\in\text{even}}}^{\infty} \sum_{i_1,...,i_n=1}^{N} \{\mathbf{T}_{i_1}^{A_1}...\mathbf{T}_{i_n}^{A_n}\} D_{A_1...A_n}^{n;i_1...i_n}(\{v\}).$$

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In turn, $\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\},\epsilon^*)$, SAD has only even color-multipoles

$$\boldsymbol{\gamma}_s(\{v\}) \quad = \quad \sum_{\substack{n=2\\ n\in\text{even}}}^{\infty} \sum_{i_1,...,i_n=1}^{N} \{\mathbf{T}_{i_1}^{A_1}...\mathbf{T}_{i_n}^{A_n}\} \gamma_{A_1...A_n}^{n;i_1...i_n}(\{v\}).$$

Absence of tri-pole is known [Aybat, et al,0607309;Dixon, et al, 0910.3653]

- Quadrupole arises at 3-loops
- Sextupole arises at 5-loops
- etc.

Conclusion

I believe that there are other (not yet explored) consequences.

done TMD soft factor for SIDIS = TMD soft factor for DY (universality)

- Duality of TMD soft factor
- Possible lattice applications

Limitations

- T-ordered operator
- Unrestricted phase space

ϵ^* method is a very powerful tool

- 3-loop evolution kernel for a twist-2 string operator (utilizing 3-loop DGLAP anomalous dimension) [Braun, et al,]
- BK/BMS relation at sub-leading orders [S.Caron-Huot,talk at HEP]
- Matching coefficient functions for TMD operators
- ϵ^* can be used as a summation prescription

Conclusion

- The rapidity divergences are alike UV divergences (in CFT)
- Renormalization theorem for rapidity divergences
- RAD/SAD correspondence (checked up to three-loop order for N=2 case (TMD), checked up to two-loop order for general case)
- Three-loop general rapidity anomalous dimension, and all order constraints on SAD
- Factorization for double-Drell-Yan (multi-Drell-Yan)

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