Geometrical structure of soft factors & rapidity renormalization theorem

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based on [1707.07606]

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General structure of the factorization theorems

The modern factorization theorems have the following general structure

$$\underbrace{\frac{d\sigma}{dX}}_{cross-X} = \underbrace{H}_{\substack{\text{Hard part}\\\text{perturbative}}} \times \underbrace{f_1 \otimes ... \otimes J_2}_{\substack{\text{Parton distributions}\\\text{jet-functions, etc}\\\text{Non-pertrubative}\\\text{universal}}} \times \underbrace{S}_{\substack{\text{Soft factor(s)}\\\text{perturbative ?}}} + \substack{\text{Some power}\\\text{suppressed terms}}$$

- Individual terms in the product are singular, and requires redefinition/refactorization
- The next factorization step is to factorize the divergences in soft factors
- The SF factorization is essential for low*er*-energy studies (e.g. SIDIS) and for resummation

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> In this talk, I will present **the last ingredient** of TMD-like factorization theorems, namely, **the factorization of rapidity divergences** and some of its consequences

- Currently, it the only proof of rapidity divergences factorization
- It is unusual. It is build on the conformal transformation and mapping of divergences.
- The proof is made for the multi-Drell-Yan process (see [talk of M.Diehl]) for arbitrary number of particles.
- It allows a number of non-trivial predictions and consequences.

Reminder
TMD factorization

TMD factorization

TMD factorization ($Q^2 \gg q_T^2$) gives us the following expression

$$\frac{d\sigma}{dQ dy\, d^2 q_T} \sim \int d^4x\, e^{iqx} \sum_X \langle h_1 | J^\mu(x) | X; h_2 \rangle \langle X; h_2 | J^\nu(0) | h_1 \rangle$$

TMD factorization

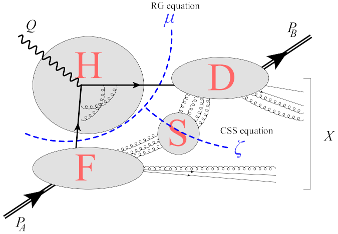
$$\frac{d\sigma}{dQ dy\, d^2 q_T} \sim \int d^2 b_T\, e^{-i(qb)_T} H(Q^2)\ \ \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

TMD soft factor (very singular)

power suppressed terms

TMD FF (singular)

TMD PDF (singular)



All components of factorization formula contain rapidity divergences.
Within soft factor rapidity divergencs entangle PDF and FF

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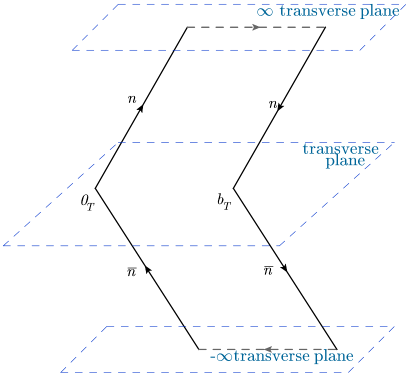
$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



Light-like vectors:

$$n^2 = \bar{n}^2 = 0, \qquad (n\cdot\bar{n}) = 1$$

Wilson line (ray)

$$\mathbf{\Phi}_v(x) = P\exp\left(ig\int_0^\infty d\sigma v^\mu A_\mu^A(v\sigma + x)\mathbf{T}^A\right)$$

Multiple divergences!

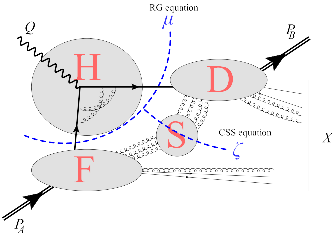
- Ultraviolet (renormalize)
- Collinear & mass (cancel in sum)
- Rapidity

Assumption: $\exp(A\ln(\delta^+\delta^-)+B)=\exp\left(\frac{A}{2}\ln((\delta^+)^2\zeta)+\frac{B}{2}\right)\exp\left(\frac{A}{2}\ln((\delta^-)^2\zeta^{-1})+\frac{B}{2}\right)$

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;\;\Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T)+Y$$

spliting rapidity singularities
$S(b_T)\to\sqrt{S(b_T;\zeta^+)}\sqrt{S(b_T;\zeta^-)}$

$$d\sigma\sim\int d^2b_T\,e^{-i(qb)_T}H(Q^2)\;F(z_1,b_T;\zeta^+)\;D(z_2,b_T;\zeta^-)+Y$$



TMD PDF
$\sqrt{S}\Phi_{h_1}$
(regular)

TMD FF
$\sqrt{S}\Delta_{h_2}$
(regular)

The extra "factorization" introduces extra scale ζ.
And corresponded evolution equation

$$\zeta\frac{d}{d\zeta}F=\frac{A}{2}F=-\mathcal{D}F$$

Rapidity anomalous dimension (RAD)

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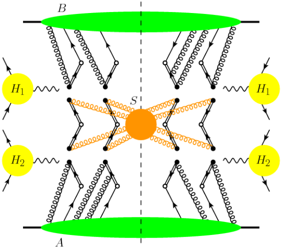
DPD soft factor (very singular)

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\; F_{h1}^{A}(z_{1,2}, b_{1,2,3,4}) S^{AB}(b_{1,2,3,4}) \bar{F}_{h2}^{B}(z_{1,2}, b_{1,2,3,4}) + Y$$

power suppressed terms

DPD (singular)



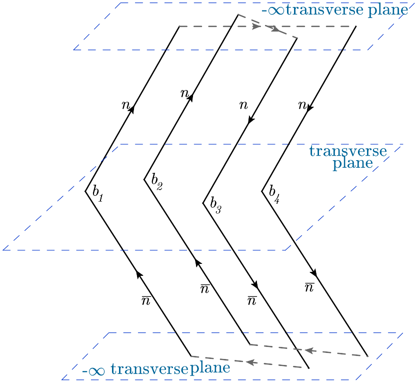
pictures from [1510.08696]

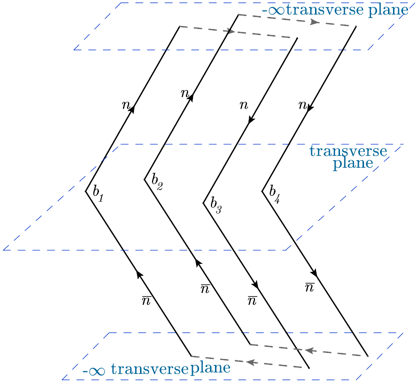
Structure is similar to TMD Drell-Yan
but now it contains
COLOR
The soft factor is a matrix

Color structure makes a lot of difference

$$F_{h1}^{A} S^{AB} \bar{F}_{h2}^{B} \xrightarrow{\text{singlets}} (F^{\mathbf{1}}, F^{\mathbf{8}}) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar{F}^{\mathbf{1}} \\ \bar{F}^{\mathbf{8}} \end{pmatrix}$$

- Soft-factors $S^{\mathbf{ij}}$ are sum of Wilson loops and double Wilson loops (all possible connections).
- Soft-factors are non-zero even in the integrated case.





Finalizing DPD factorization

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}] H_1(Q_1^2) H_2(Q_2^2)\ \left(F^{\mathbf{1}}, F^{\mathbf{8}}\right) \begin{pmatrix} S^{\mathbf{11}} & S^{\mathbf{18}} \\ S^{\mathbf{81}} & S^{\mathbf{88}} \end{pmatrix} \begin{pmatrix} \bar F^{\mathbf{1}} \\ \bar F^{\mathbf{8}} \end{pmatrix} + Y$$

spliting rapidity singularities
$S(b_{1,2,3,4}) \to s^T(b_{1,2,3,4};\zeta^+) s(b_{1,2,3,4};\zeta^-)$
(possible at NNLO [AV,1608.04920])

$$\frac{d\sigma}{dX} \sim \int [db_T\, e^{-i(qb)_T}]\, e^{-i(qb)_T} H_1(Q_1^2) H_2(Q_2^2)\ F(z_{1,2}, b_{1,2,3,4};\zeta^+) \bar F(z_{1,2}, b_{1,2,3,4};\zeta^-) + Y$$

DPD $(sF_{h_1})^T$ (regular)

DPD sF_{h_2} (regular)

Matrix rapidity evolution

$$\frac{dF(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)}{d\ln\zeta} = -F(z_{1,2}, \mathbf{b}_{1,2,3,4};\zeta,\mu)\mathbf{D}(\mathbf{b}_{1,2,3,4},\mu)$$

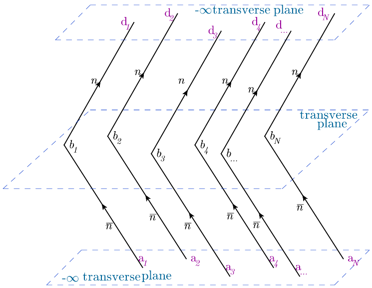
where $\mathbf{D}$ is matrix build (linearly) of TMD rapidity anomalous dimensions. E.g.

$$D^{\mathbf{18}} = \frac{\mathcal{D}(\mathbf{b}_{12}) - \mathcal{D}(\mathbf{b}_{13}) - \mathcal{D}(\mathbf{b}_{24}) + \mathcal{D}(\mathbf{b}_{34})}{\sqrt{N_c^2-1}}$$

The same story for multi-parton scattering

- Just as double-parton, but multi..(four WL's → arbitrary number WL's)
- The soft-factor has many Wilson lines [M.Diehl,D.Ostermeier,A.Schafer,1111.0910]

$$\mathbf{\Sigma}(\{b\}) = \langle 0|T\{[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_N)\dots[\mathbf{\Phi}_{-n}\mathbf{\Phi}^{\dagger}_{-\bar{n}}](b_1)\}|0\rangle$$



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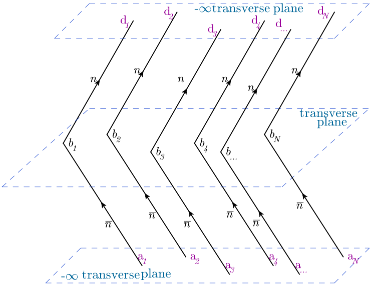

Result at NNLO is amazingly simple

$$\mathbf{\Sigma}(\mathbf{b}_{1,..,N}) = \exp\left(-\sum_{i<j}\mathbf{T}_i^A\mathbf{T}_j^A\sigma(\mathbf{b}_{ij}) + \mathcal{O}(a_s^3)\right)$$

- σ – TMD soft factor
- $\mathbf{T}_i^A\mathbf{T}_j^A$ = "dipole"
- $\mathcal{O}(a_s^3)$ contains also "color-multipole" terms
- Rapidity factorization for dipole part is straightforward (assuming TMD factorization)

The rapidity factorization is shown at 2-loops, but not proved.

- Proof is required to define universal non-perturbative functions.
- There are possible issues at 3-loops, due to 4-WL interactions.
- To make a proof we have to understand the structure of divergences of SFs.

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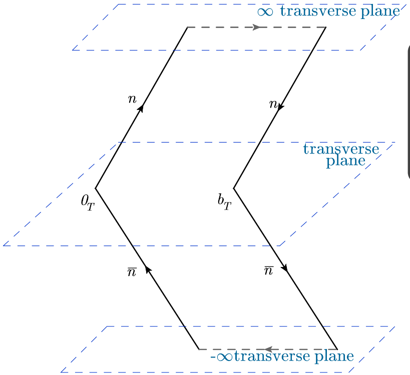
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\text{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$



Although traditionally the diagrams are considered in the momentum space, the coordinate representation is more natural and clean.

- Ultraviolet (small-distances)
- Collinear & mass (large distances)
- Rapidity (small/large distances)

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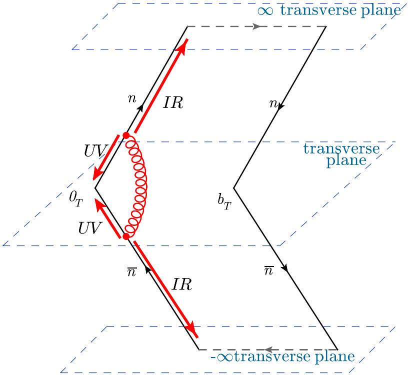
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\begin{aligned}\int \quad dxdy \quad & D(x-y)\\ = \quad & \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{x^+y^-}\\ = \quad & \int_0^\infty \frac{dx^+}{x^+}\int_0^\infty \frac{dy^-}{y^-}\\ = \quad & (\mathrm{UV}+\mathrm{IR})(\mathrm{UV}+\mathrm{IR})\end{aligned}$$

Some people set it to zero.

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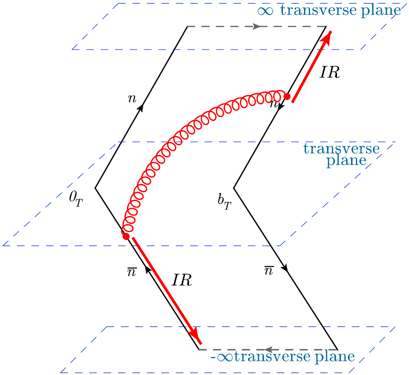
TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}_{-\bar{n}}^\dagger(\mathbf{0}_T)\right)|0\rangle$$

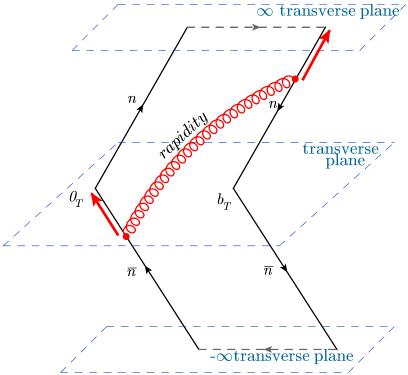


$$\int dxdy \quad D(x-y)$$
$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$
$$= \text{IR at } x, y \to \infty$$

However, it exactly cancels IR from the previous diagram
Proved at all orders,
e.g.[Echevarria,Scimemi,AV,1511.05590]

TMD soft factor

$$S(\mathbf{b}_T) = \langle 0|\mathrm{Tr}\left(\mathbf{\Phi}_n(\mathbf{0}_T)\mathbf{\Phi}_n^\dagger(\mathbf{b}_T)\mathbf{\Phi}_{-n}(\mathbf{b}_T)\mathbf{\Phi}^\dagger_{-\bar{n}}(\mathbf{0}_T)\right)|0\rangle$$



$$\int \quad dxdy \quad D(x-y)$$

$$= \int_0^\infty dx^+ \int_0^\infty dy^- \frac{1}{(2x^+y^- + \mathbf{b}_T^2)}$$

$$= \text{rap. div. at } \lim_{\lambda\to 0}\{x=\lambda, y=\lambda^{-1}\}$$

Rapidity divergence is a special kind of divergences, UV& IR
Does not cancel.

Deeper analysis shows that rapidity divergences are similar to UV divergences

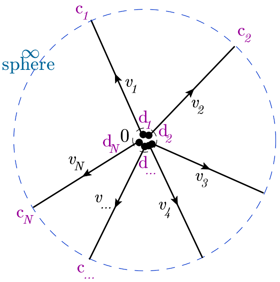
- The rapidity divergence corresponds to the radiation of gluon from the transverse "plane" to the light-cone infinity.
- The counting rule for rapidity divergences is topologically (on the level of graphs) similar to counting of UV divergences.
- See the detailed analysis in [1707.07606].

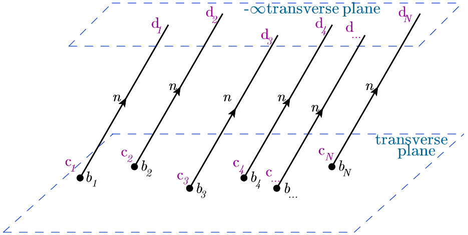
Rapidity divergences associated with transverse planes (or better to say with the layer between the transverse plane and infinity). Let us construct a conformal transformation which maps it to the point (to a sphere).

Conformal-stereographic transformation

$$\mathcal{C}_{\bar{n}} : \{x^+, x^-, x_\perp\} \to \{\frac{-1}{2a}\frac{1}{\lambda + 2ax^+}, x^- + \frac{ax_\perp^2}{\lambda + 2ax^+}, \frac{x_\perp}{\lambda + 2ax^+}\}$$

- Translation – special conformal transformation (along n) – Translation
- a and λ are free parameters





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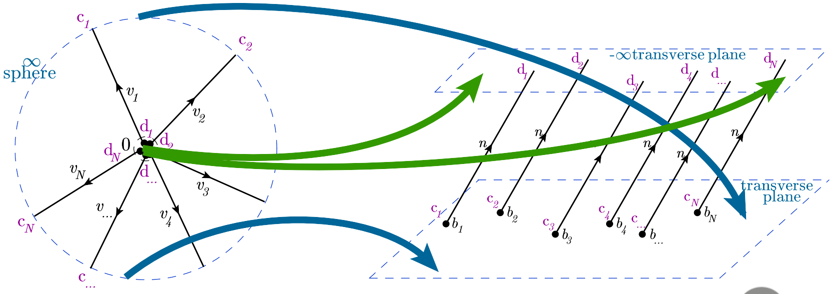

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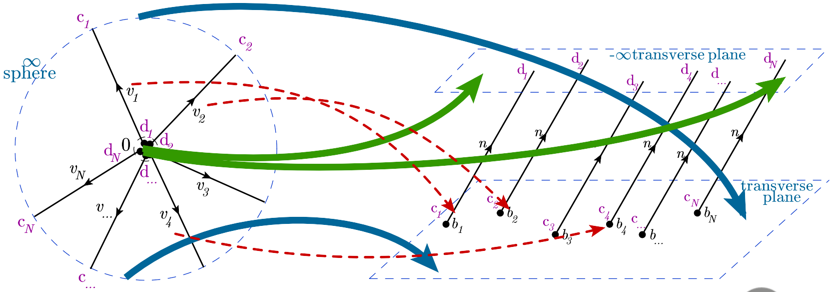

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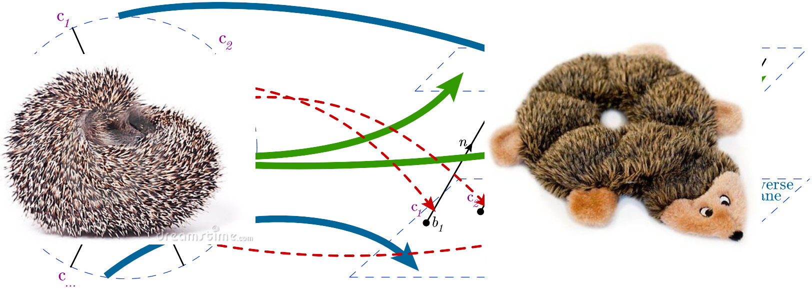


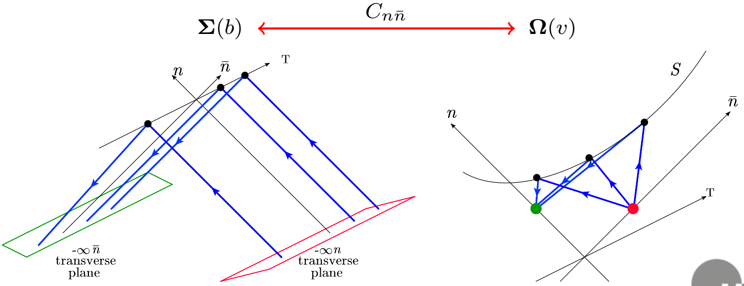
Composition of two conformal-stereographic transformations

$$C_{n\bar{n}} = \mathcal{C}_n \mathcal{C}_{\bar{n}} = \mathcal{C}_{\bar{n}} \mathcal{C}_n$$

With the special choice of parameters

$$a\lambda < 0, \qquad \bar{a}\bar{\lambda} < 0, \qquad (a\bar{a})^2 < \frac{1}{2\rho_T^2} \qquad \rho_T^2 > \max\{b_i^2\},$$

any DY-like soft factor transforms to a compact object.



In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization

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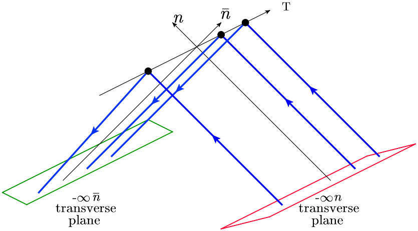
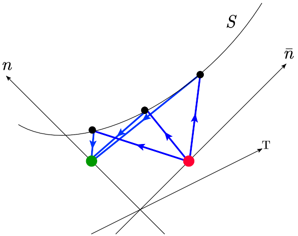
In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization

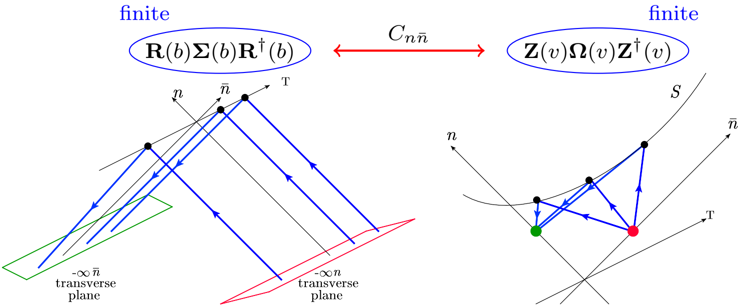
$$\mathbf{R}(b)\mathbf{\Sigma}(b)\mathbf{R}^{\dagger}(b) \xleftrightarrow{C_{n\bar{n}}} \mathbf{Z}(v)\mathbf{\Omega}(v)\mathbf{Z}^{\dagger}(v)$$





In conformal QFT rapidity divergences equivalent to UV divergences

- The UV renormalization imposes rapidity divergence renormalization
- There are also UV renormalization factors in cusps (we omit them for a moment)



RDRT in conformal theory

In a conformal theory rapidity divergences can be removed (*renormalized*) by a multiplicative factor.

$$C_{n\bar{n}}^{-1}\left(\mathbf{Z}(\{v\},\mu)\right) = \mathbf{R}_n(\{b\},\nu^+)$$

Rapidity anomalous dimension (RAD)

$$\mathbf{D}(\{b\}) = \frac{1}{2}\mathbf{R}_n^{-1}(\{b\},\nu^+)\nu^+\frac{d}{d\nu^+}\mathbf{R}_n(\{b\},\nu^+),$$

In CSS notation it is $-K$, in [Becher,Neubert] $F_{q\bar{q}}$, in SCET literature γ_ν.

(In CFT) DY-like Soft factors expresses as

$$\mathbf{\Sigma}(\{b\},\delta^+,\delta^-) \quad = \quad e^{2\mathbf{D}(\{b\})\ln(\delta^+/\nu^+)}\overbrace{\mathbf{\Sigma}_0(\{b\},\nu^2)}^{\text{finite}}e^{2\mathbf{D}^\dagger(\{b\})\ln(\delta^-/\nu^-)},$$

From conformal theory to QCD

QCD at the critical point

QCD is conformal in $4-2\epsilon^*$ dimensions

$$\beta(\epsilon^*)=0, \qquad \Rightarrow \qquad \epsilon^*=-a_s\beta_0-a_s^2\beta_1-....$$

From conformal theory to QCD

QCD at the critical point

QCD is conformal in $4-2\epsilon^*$ dimensions

$$\beta(\epsilon^*)=0, \qquad \Rightarrow \qquad \epsilon^*=-a_s\beta_0-a_s^2\beta_1-....$$

Thus, at $4-2\epsilon^*$ dimensions, the rapidity renormalization theorem works.

- Starting from the leading conformal invariant term, one proves by induction

$$\mathbf{\Sigma}(\{b\},\delta^+,\delta^-) \quad = \quad e^{2\mathbf{D}(\{b\})\ln(\delta^+/\nu^+)}\overbrace{\mathbf{\Sigma}_0(\{b\},\nu^2)}^{\text{finite}}e^{2\mathbf{D}^\dagger(\{b\})\ln(\delta^-/\nu^-)},$$

$$\mathbf{D}_{\text{QCD}}\neq\mathbf{D}_{\text{CFT}}$$

Main conclusion

The rapidity divergences in TMD-like soft factors can be renormalized at any (finite) order of perturbation theory. It is equivalent to proof of factorization of rapidity divergences.

Many consequences

- Factorization for multi-Drell-Yan process.
- **Correspondence between soft and rapidity anomalous dimensions.**
- **N^3LO rapidity anomalous dimension (for "free").**
- Constraints of soft anomalous dimension.
- Universality of DY and SIDIS TMD soft factors
- Absence of (naive) factorization for particular processes.
- Many others, (yet unexplored).

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Soft/rapidity anomalous dimension correspondence

The equivalence (under conformal transformation) between **Z** and **R** implies the equality between corresponding anomalous dimensions

$$\gamma_s(\{v\}) = 2\mathbf{D}(\{b\})$$

It has been observed in [Li,Zhu,1604.01404].

- UV anomalous dimension independent on ϵ
- Rapidity anomalous dimension does depend on ϵ
- At ϵ^* conformal symmetry of QCD is restored

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In QCD

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

- Exact relation!
- Connects different regimes of QCD

→ Lets test it.

$$\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$$

How to use it?

- Physical value is $\mathbf{D}(\{\mathbf{b}\}, 0)$
- $\epsilon^* = 0 - a_s\beta_0 - a_s^2\beta_1 - a_s^3\beta_2 - ....$
- We can compare order by order in PT

$$\begin{aligned}
\mathbf{D}_1(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_1(\{v\}), \\
\mathbf{D}_2(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_2(\{v\}) + \beta_0\mathbf{D}_1'(\{b\}), \\
\mathbf{D}_3(\{b\}) &= \frac{1}{2}\boldsymbol{\gamma}_3(\{v\}) + \beta_0\mathbf{D}_2'(\{b\}) + \beta_1\mathbf{D}_1'(\{b\}) - \frac{\beta_0^2}{2}\mathbf{D}_1''(\{b\}),
\end{aligned}$$

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TMD rapidity anomalous dimension

- $N=2$, no matrix structure, $\boldsymbol{B}=\frac{b^2}{4}$, $L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow\ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$ NLO | NLO $-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}$ $a_s\mathcal{D}^{(1)}$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{\nu^2}\right)+0\right)=a_s\left(\ln\left(\frac{\mathbf{b}^2\mu^2}{4e^{-2\gamma_E}}\right)+0\right)$$

Obvious relation, QCD is conformal at leading order.

$\nu^2=4e^{-2\gamma_E}$

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- $N=2$, no matrix structure, $\boldsymbol{B}=\frac{b^2}{4}$, $L=\ln\left(\frac{\boldsymbol{B}\mu^2}{e^{-2\gamma_E}}\right)\leftrightarrow\ln\left(\frac{v_{12}\mu^2}{\nu^2}\right)$

$\gamma_s^{(1)}=L+0$

$\gamma_s^{(2)}=[(\frac{67}{9}-2\zeta_2)C_A-\frac{20}{18}N_f]L+$
$(28\zeta_3+\dots)C_A+(\frac{112}{27}-\frac{4}{3}\zeta_2)N_f$

NLO | NLO

N^{2}LO

$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\frac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$

$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}$ $\quad$ $a_s\mathcal{D}^{(1)}+a_s^2 ?\mathcal{D}^{(2)}$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$..+a_s^2\left(\Gamma_2 L+\gamma^{(2)}\right)=..+2a_s^2\left(\mathcal{D}^{(2)}-2\beta_0(L^2+\zeta_2)\right)$$

We found 2-loop rapidity anomalous dimension

$$\mathcal{D}^{(2)}_{L=0}=\left(\frac{404}{27}-14\zeta_3\right)C_A-\frac{112}{27}\frac{N_f}{2}$$

TMD rapidity anomalous dimension

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$$\gamma_s^{(1)}=L+0$$

$$\gamma_s^{(2)}=[(\tfrac{67}{9}-2\zeta_2)C_A-\tfrac{20}{18}N_f]L+(28\zeta_3+...)C_A+(\tfrac{112}{27}-\tfrac{4}{3}\zeta_2)N_f$$

$$\gamma_s^{(3)}=[\tfrac{245}{3}C_A^2+...]L++(-192\zeta_5C_A^2+...+\tfrac{2080}{729}N_f^2)$$

NLO | NLO

N^{2}LO | N^{2}LO

N^{3}LO

$$-2\left(\boldsymbol{B}^\epsilon\Gamma(-\epsilon)+\tfrac{1}{\epsilon}\right)=\mathcal{D}^{(1)}$$

$$\boldsymbol{B}^{2\epsilon}\Gamma^2(-\epsilon)\Big(C_A(2\psi_{-2\epsilon}-2\psi_{-\epsilon}+\psi_\epsilon+\gamma_E)=\mathcal{D}^{(2)}+\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)}\Big(\frac{3(4-3\epsilon)}{2\epsilon}C_A-N_f\Big)\Big)+\boldsymbol{B}^\epsilon\frac{\Gamma(-\epsilon)}{\epsilon}\beta_0+\frac{\beta_0}{2\epsilon^2}-\frac{\Gamma_1}{2\epsilon}$$

[Echevarria,Scimemi,AV,1511.05590]

$$a_s\gamma_s^{(1)}+a_s^2\gamma_s^{(2)}+a_s^3\gamma_s^{(3)} \qquad a_s\mathcal{D}^{(1)}+a_s^2\mathcal{D}^{(2)}+a_s^3?\mathcal{D}^{(3)}$$

$$\gamma_s(\{v\})=2\mathcal{D}(\{\mathbf{b}\},\epsilon^*)$$

Expand in a_s

$$..+a_s^3\left(\Gamma_3L+\gamma^{(3)}\right)=..+2a_s^2\Big[\mathcal{D}^{(3)}-\frac{2\beta_0^2}{3}L^3-\left(\frac{\beta_0\Gamma_1}{2}+\beta_1\right)L^2+\beta_0(\gamma_1-2\beta_0\zeta_2)L-\beta_0\Gamma_1\frac{\zeta_2}{4}-\zeta_2\beta_1+\frac{2\beta_0^2}{3}\left(\zeta_3-\frac{82}{9}\right)+26\beta_0C_A\left(\zeta_4-\frac{8}{27}\right)\Big]$$

We found 3-loop rapidity anomalous dimension

$$\mathcal{D}^{(3)}_{L=0}=C_A^2\left(\frac{297029}{1458}+\frac{88}{3}\zeta_2\zeta_3+...+96\zeta_5\right)+...+C_FN_f\left(\frac{-152}{9}\zeta_3-8\zeta_4+\frac{11711}{54}\right)$$

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$$\begin{aligned}\mathcal{D}^{(3)}_{L=0} = & -\frac{C_A^2}{2}\left(\frac{12328}{27}\zeta_3 - \frac{88}{3}\zeta_2\zeta_3 - 192\zeta_5 - \frac{297029}{729} + \frac{6392}{81}\zeta_2 + \frac{154}{3}\zeta_4\right) \\ & -\frac{C_A N_f}{2}\left(-\frac{904}{27}\zeta_3 + \frac{62626}{729} - \frac{824}{81}\zeta_2 + \frac{20}{3}\zeta_4\right) - \\ & \frac{C_F N_f}{2}\left(-\frac{304}{9}\zeta_3 + \frac{1711}{27} - 16\zeta_4\right) - \frac{N_f^2}{2}\left(-\frac{32}{9}\zeta_3 - \frac{1856}{729}\right)\end{aligned}$$

- Coincides with the one calculated directly [Li,Zhu,1604.01404]
- The logarithmic structure of rapidity anomalous dimension also restored

$$\mu^2\frac{d}{d\mu^2}\mathcal{D}(a_s(\mu),\mathbf{b}) = \frac{\Gamma_{cusp}(a_s(\mu))}{2}$$

vs.

$$\nu^2\frac{d}{d\nu^2}\gamma_s(\nu,v) = \frac{\Gamma_{cusp}}{2}$$

UV anomalous dimensions independent on ϵ. UV anomalous dimension of rapidity anomalous dimension also.

Quadrupole part of SAD

$$\boldsymbol{D}(\{b\}) = -\frac{1}{2}\sum_{[i,j]}\mathbf{T}_i^A\mathbf{T}_j^A\mathcal{D}_{\text{TMD}}(b_{ij}) - \sum_{[i,j,k,l]} if^{ACE}if^{EBD}\mathbf{T}_i^A\mathbf{T}_j^B\mathbf{T}_k^C\mathbf{T}_l^D\mathcal{F}_{ijkl} - \sum_{[i,j,k]}\mathbf{T}_i^{\{AB\}}\mathbf{T}_j^C\mathbf{T}_k^D if^{ACE}if^{EBD}C + \mathcal{O}(a_s^4),$$

Quadrupole part has been calculated in [Almelid,Duhr,Gardi;1507.00047]

$$\tilde{C} = a_s^3\left(\zeta_2\zeta_3 + \frac{\zeta_5}{2}\right) + \mathcal{O}(a_s^4),$$

$$\tilde{\mathcal{F}}_{ijkl}(\{b\}) = 8a_s^3\mathcal{F}(\tilde{\rho}_{ikjl},\tilde{\rho}_{iljk}) + \mathcal{O}(a_s^4),$$

$$\rho_{ijkl} = \frac{(v_i\cdot v_j)(v_k\cdot v_l)}{(v_i\cdot v_k)(v_j\cdot v_l)} \quad\leftrightarrow\quad \tilde{\rho}_{ijkl} = \frac{(b_i-b_j)^2(b_k-b_l)^2}{(b_i-b_k)^2(b_j-b_l)^2}$$

Matrix CS equation

$$\mu^2\frac{d\mathbf{D}(\mu,\{b\})}{d\mu^2} = \sum_{i=1}^{N}\frac{\Gamma^i_{cusp}}{4}\mathbf{I},$$

Main conclusion

The rapidity divergences in TMD-like soft factors can be renormalized at any (finite) order of perturbation theory. It is equivalent to proof of factorization of rapidity divergences.

Many consequences

- Factorization for multi-Drell-Yan process.
- Evolution for multiPDs (at 3-loops).
- **Correspondence between soft and rapidity anomalous dimensions.**
- **N^3LO rapidity anomalous dimension (for "free").**
- Constraints of soft anomalous dimension.
- Universality of DY and SIDIS TMD soft factors
- Absence of (naive) factorization for particular processes.
- Many others, (yet unexplored).

Example then it does not work (no factorization?)

There are talks about "dipole-like" distributions that could appear in processes like $pp \to hX$ e.g. [Boer,et al,1607.01654]

However, it is straightforward to show that the factorization is necessarily broken (or have not closed form)

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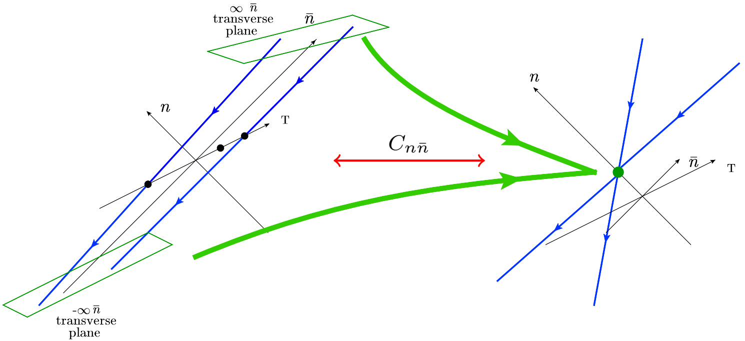

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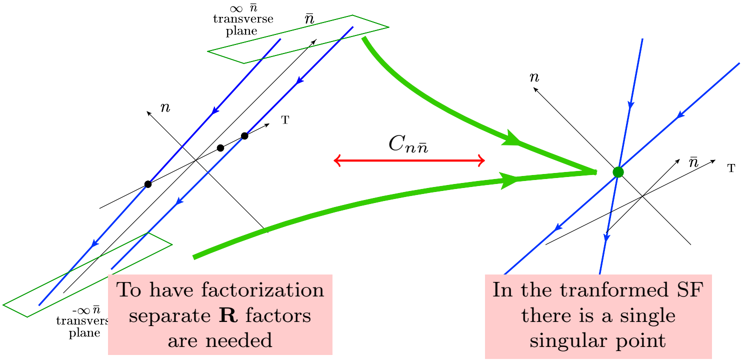

- The renormalization of dipole recouple colors → extra gauge link → ala BK equation.

Quadrupole part of SAD

$$\boldsymbol{\gamma}_s(\{v\}) = -\frac{1}{2}\sum_{[i,j]} \mathbf{T}_i^A \mathbf{T}_j^A \gamma_{\text{dipole}}(v_i \cdot v_j) - \sum_{[i,j,k,l]} if^{ACE} if^{EBD} \mathbf{T}_i^A \mathbf{T}_j^B \mathbf{T}_k^C \mathbf{T}_l^D \mathcal{F}_{ijkl}$$
$$- \sum_{[i,j,k]} \mathbf{T}_i^{\{AB\}} \mathbf{T}_j^C \mathbf{T}_k^D if^{ACE} if^{EBD} C + \mathcal{O}(a_s^4),$$

Quadrupole part has been calculated in [Almelid,Duhr,Gardi;1507.00047]

$$\tilde{C} = a_s^3 \left(\zeta_2\zeta_3 + \frac{\zeta_5}{2}\right) + \mathcal{O}(a_s^4),$$
$$\tilde{\mathcal{F}}_{ijkl}(\{b\}) = 8a_s^3 \mathcal{F}(\tilde{\rho}_{ikjl}, \tilde{\rho}_{iljk}) + \mathcal{O}(a_s^4),$$

Quadrupole part of RAD

- Color structures are not affected by ϵ^*
- Quadrupole contribution depends only on conformal ratios

$$\rho_{ijkl} = \frac{(v_i \cdot v_j)(v_k \cdot v_l)}{(v_i \cdot v_k)(v_j \cdot v_l)} \quad \leftrightarrow \quad \tilde{\rho}_{ijkl} = \frac{(b_i - b_j)^2 (b_k - b_l)^2}{(b_i - b_k)^2 (b_j - b_l)^2}$$

The correspondence between SAD and RAD can be used also to constraint the SAD. It seems that structure of RAD (diagrammatically) is simpler.

Color-structure of soft anomalous dimension

As a consequence of Lorentz invariance one has

$$\mathbf{\Sigma}(\{b\}) = \mathbf{\Sigma}^\dagger(\{b\})$$

It implies that RAD has only even color-multipoles

$$\mathbf{D}(\{b\}) \quad = \quad \sum_{\substack{n=2 \\ n\in\text{even}}}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \{\mathbf{T}_{i_1}^{A_1}...\mathbf{T}_{i_n}^{A_n}\} D_{A_1\ldots A_n}^{n;i_1\ldots i_n}(\{v\}).$$

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In turn, $\boldsymbol{\gamma}_s(\{v\}) = 2\mathbf{D}(\{\mathbf{b}\}, \epsilon^*)$, SAD has only even color-multipoles

$$\boldsymbol{\gamma}_s(\{v\}) = \sum_{\substack{n=2 \\ n\in\text{even}}}^{\infty} \sum_{i_1,...,i_n=1}^{N} \{\mathbf{T}_{i_1}^{A_1}...\mathbf{T}_{i_n}^{A_n}\} \gamma_{A_1...A_n}^{n;i_1...i_n}(\{v\}).$$

Absence of tri-pole is known [Aybat, et al,0607309;Dixon, et al, 0910.3653]

- Quadrupole arises at 3-loops
- Sextupole arises at 5-loops
- etc.